BASES PROPERTIES OF SEQUENCES OF DILATED PERIODIC FUNCTIONS IN BANACH AND HILBERT SPACES

by

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Abstract

The main aim of the thesis is to continue the investigations as to which extent the family of dilations $\mathfrak{E}_f := \{f_n\}_n$, where $f : \mathbb{R} \longrightarrow \mathbb{C}$ and $f_n(\cdot) := f(n\cdot)$ for all $n \in \mathbb{N}$, forms a basis of $L^r(0,1)$ for $r \in (1, \infty)$.

We introduce an improved one-term and new multi-term criteria for determining Schauder and Riesz bases properties of the family $\mathfrak{E}_f$ in the context of Lebesgue spaces. We develop the concept of multipliers on Hardy spaces of polydiscs and establish an analogy to the preceding criteria in this setting. We illustrate the rich structure behind this problem by applying these criteria to various families of generalised (p,q)-trigonometric functions, such as, the p-cosine, the p-sine, the p-exponential and the (p,q)-cosine functions. These functions arise naturally in the study of eigenspaces of the one-dimensional Dirichlet problem for the (p,q)-Laplacian. The approach was proved fruitful and the findings achieved follow naturally from previously known results.

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Contents

Basic Notations

Let $p, q, r \in (1, \infty)$. Throughout the thesis we will consider the following notations:

$\mathbb{N}$: Set of all natural numbers (positive integers)
$\mathbb{P}(\mathbb{N})$: Set of all prime numbers not including 1
$\mathcal{F}$: Finite subset of $\mathbb{N}$ such that $1 \in \mathcal{F}$
$\mathbb{P}(\mathcal{F})$: Set of all primes in $\mathbb{P}(\mathbb{N})$ dividing $n \in \mathcal{F}$
$\mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$
$\mathbb{N}^n$: Set of all $\nu = (\nu_1, \ldots, \nu_n)$ with $\nu_j \in \mathbb{N}$
$\mathbb{N}^\infty$: Set of all $\nu = (\nu_1, \nu_2, \ldots)$ with $\nu_j \in \mathbb{N}$ for $j \in \mathbb{N}$
$\mathbb{N}_0^\infty$: Set of all $\nu \in \mathbb{N}^\infty$ with $\nu_j \neq 0$ for finitely many j
$\mathbb{Z}$: Set of all integers
$\mathbb{Z}^n, \mathbb{Z}^\infty$ and $\mathbb{Z}_0^\infty$: Sets analogous to $\mathbb{N}^n, \mathbb{N}^\infty$ and $\mathbb{N}_0^\infty$ respectively
$\mathbb{R}$: Set of all real numbers
$\mathbb{C}$: Set of all complex numbers
$\mathbb{C}^n, \mathbb{C}^\infty$: Cartesian products of n and infinitely many copies of $\mathbb{C}$
$\mathbb{D}$: Open unit disc
$\mathbb{D}^n, \mathbb{D}^\infty$: Finite and infinite dimensional polydiscs
$\mathbb{T}$: Unit circle
$\mathbb{T}^n, \mathbb{T}^\infty$: Finite and infinite dimensional torus
L^r: Lebesgue spaces for $r \in (1, \infty]$
H^r: Hardy spaces for $r \in (1, \infty]$
$\mathfrak{E}_f := \{f_n(\mathfrak{e}(f)\cdot)\}_n$ where $f : \mathbb{R} \longrightarrow \mathbb{C}$ is a periodic function possibly given in terms of a parameter $\mathfrak{e}(f) > 0$ such that $f_n(\mathfrak{e}(f)\cdot) = f(n\mathfrak{e}(f)\cdot)$
$e_j(\cdot)$: Denotes $\sin(j\pi\cdot)$ (odd) or $\cos(j\pi\cdot)$ (even) functions
$\mathbf{a} = (\mathbf{a}_j)_j$: Fourier coefficients of any $f \in L^r(0, 1)$ in terms of the basis $\mathfrak{E}_{\sin}$
$\mathbf{b} = (\mathbf{b}_j)_j$: Fourier coefficients of any $f \in L^r(0, 1)$ in terms of the basis $\mathfrak{E}_{\sin_{p,q}}$
$a_j \equiv a_j(p)$: Sine Fourier coefficients of $\sin_p(\pi_p\cdot)$
$b_j \equiv b_j(p)$: Cosine Fourier coefficients of $\cos_p(\pi_p\cdot)$
$\tau_j \equiv \tau_j(p, q)$: Sine Fourier coefficients of $\sin_{p,q}(\pi_{p,q}\cdot)$ where $\tau_j(p, p) \equiv a_j(p)$
$\eta_j \equiv \eta_j(p, q)$: Cosine Fourier coefficients of $\cos_{p,q}(\pi_{p,q}\cdot)$ where $\eta_j(p, p) \equiv b_j(p)$
$c \equiv_k d$: c is congruent to d modulo k
$c_k \lesssim d_k$: $\exists C > 0$ independent of k such that $c_k \leq C d_k$ for all $k \in \mathbb{N}$ given that $(c_k), (d_k)$ are non-negative sequences of real numbers
$\mathcal{B}(X, Y)$: Set of all bounded linear operators on X to Y with $\mathcal{B}(X) \equiv \mathcal{B}(X, X)$
Id: Identity operator

Chapter 1

Introduction

1.1 Generalised trigonometric functions

Certain generalisations of the classical trigonometric functions have attracted much interest in recent years. To explain what these are, let $p \in (1, \infty)$ and define $F_p : [0,1] \longrightarrow [0, \pi_p/2]$ by

$$F_p(x) = \int_0^x (1 - t^p)^{-1/p} \mathrm{d}t.$$

The inverse of this function is denoted by $\sin_p$. Initially this is defined on the interval $[0, \pi_p/2]$, where $\pi_p = 2F_p(1)$, but a process of extension by symmetry and periodicity leads to a function, also denoted by $\sin_p$, defined on the whole real line, which coincides with the familiar sine function when $p = 2$.

The study of the generalised trigonometric functions has a long history. They appeared in some form in the work of Levin [27], then more systematically in Elbert [16], Ôtani [33] and Lindqvist [29]. The popularity of these functions stems from its connection with the one-dimensional p-Laplacian: for example, the Dirichlet problem

$$-\Delta_p u := -(u'|u'|^{p-2})' = \lambda u|u|^{p-2} \quad \text{on} \quad (0, \pi_p)$$

where $u(0) = u(\pi_p) = 0$, has eigenvalues $\lambda_n = (p-1)n^p$ and associated eigenvectors $\sin_p(nx), n \in \mathbb{N}$. Of course, it is not expected that these generalisations will have all the properties of their classical counterparts: they are not infinitely differentiable and do not appear to satisfy sensible addition formulae.

Further generalisations have been introduced by means of the function $F_{p,q} : [0,1] \longrightarrow [0, \pi_{p,q}/2]$ given by (1.2) below. Here p and q are arbitrary numbers in $(1, \infty)$. This leads to the definition of the function $\sin_{p,q}$ that coincides with $\sin_p$ when $p = q$ and is connected with the Dirichlet problem for the (p, q)-Laplacian:

$$-\Delta_{p,q} u := -(u'|u'|^{p-2})' = \lambda u|u|^{q-2} \quad \text{on} \quad (0, \pi_{p,q}),$$

where $u(0) = u(\pi_{p,q}) = 0$ and $p, q \in (1, \infty)$. The solutions of this equation are the eigenfunctions which are expressible in terms of the $\sin_{p,q}(nx)$ with associated eigenvalues $\lambda_n = \frac{q(p-1)}{p} n^q$ for $n \in \mathbb{N}$ [14], where $\pi_{p,q}$ is defined in (1.3) below.

In addition, a counterpart to the formula $\cos^2(x) + \sin^2(x) = 1$ was discovered in [28], which established a bridge between the p-sine and the p-cosine functions defined in (1.7). For background information about these functions and their connection with boundary-value problems we refer to [14] and [30]. We also must mention the paper of Lindqvist and Peetre [31] in which related functions were defined, the connection being that

$$\begin{aligned} &S_{1/p}(x) := \sin_{p,p'}(x), \\ &C_{1/p}(x) := |\cos_{p,p'}(x)|^{p-2} \cos_{p,p'}(x) = |\cos_{p,p'}(x)|^{p-1} \operatorname{sgn}(\cos_{p,p'}(x)), \qquad x \in \mathbb{R}, \end{aligned} \tag{1.1}$$

where $p' = p/(p-1)$ and sgn is the sign function.

1.1.1 Definitions and properties

Throughout we shall assume that $p, q \in (1, \infty)$ and use the notation $p' := p/(p-1)$.

Define the function

$$F_{p,q}(x) := \int_0^x (1 - t^q)^{-1/p} \mathrm{d}t, \qquad x \in [0, 1]. \tag{1.2}$$

Since this is strictly increasing, it has an inverse which we denote by $\sin_{p,q}$, $\sin_{p,q} :=$

$(F_{p,q})^{-1}$, to emphasise the connection with the usual sine function (note that $F_{2,2} = \sin^{-1}$). The function $\sin_{p,q}$ is defined on the interval $[0, \pi_{p,q}/2]$, where

$$\pi_{p,q} := 2F_{p,q}(1), \tag{1.3}$$

(see [14, Chapter 2] and [11]) which can also be written as

$$\pi_{p,q} = \frac{2B(1/p', 1/q)}{q} = \frac{2\Gamma(1/p')\Gamma(1/q)}{q\Gamma(1/p' + 1/q)}, \tag{1.4}$$

where B is the Beta function and Γ is the Gamma function.

Observing that $\sin_{p,q}(0) = 0$ and $\sin_{p,q}(\pi_{p,q}/2) = 1$, we can extend $\sin_{p,q}$ to $[0, \pi_{p,q}]$ by defining

$$\sin_{p,q}(x) = \sin_{p,q}(\pi_{p,q} - x), \qquad x \in [\pi_{p,q}/2, \pi_{p,q}]; \tag{1.5}$$

further odd extension to the segment $[0, 2\pi_{p,q}]$ is achieved around $\pi_{p,q}$ via the translation

$$\sin_{p,q}(x) = -\sin_{p,q}(2\pi_{p,q} - x), \qquad x \in [\pi_{p,q}, 2\pi_{p,q}]; \tag{1.6}$$

and finally $\sin_{p,q}$ is extended to the whole real line $\mathbb{R}$ by $2\pi_{p,q}$-periodicity. This extension is continuously differentiable on $\mathbb{R}$ and C^∞ everywhere except at the points $\{n\pi_{p,q}/2; n \in \mathbb{Z}\}$ [11]. The choice $p = q = 2$ corresponds to the standard trigonometric setting $\sin_{2,2} \equiv \sin$, $\pi_{2,2} \equiv \pi$. Here and everywhere below we write π_p instead of $\pi_{p,p}$ and $\sin_p$ instead of $\sin_{p,p}$.

Define the function $\cos_{p,q} : \mathbb{R} \longrightarrow [-1, 1]$ by

$$\cos_{p,q}(x) := \frac{\mathrm{d}}{\mathrm{d}x} \sin_{p,q}(x), \qquad \forall x \in \mathbb{R}. \tag{1.7}$$

Clearly, $\cos_{p,q}$ is even, $2\pi_{p,q}$-periodic and odd about $\pi_{p,q}/2$. If $x \in [0, \pi_{p,q}/2]$ and we put $y = \sin_{p,q}(x)$, then

$$\cos_{p,q}(x) = (1 - y^q)^{1/p} = (1 - \sin_{p,q}^q(x))^{1/p}. \tag{1.8}$$

Hence, $\cos_{p,q}$ is strictly decreasing on $[0, \pi_{p,q}/2]$, $\cos_{p,q} 0 = 1$, $\cos_{p,q}(\pi_{p,q}/2) = 0$ and

$$|\sin_{p,q}(x)|^q + |\cos_{p,q}(x)|^p = 1, \qquad \forall x \in \mathbb{R}.$$

It is remarkable to see that for the case $p = q \neq 2$ the derivative of $\cos_p$ is not $-\sin_p$. Moreover, while the extended $\sin_p$ function does belong to $C^1(\mathbb{R})$, it is far from being analytic on $\mathbb{R}$ if $p \neq 2$. This is because its second derivative at x can be seen to be

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} \sin_p(x) = \frac{\mathrm{d}}{\mathrm{d}x} \cos_p(x) = -(\sin_p(x))^{p-1}(\cos_p(x))^{2-p},$$

which is not continuous at $\frac{\pi_p}{2}$ if $p \in (2, \infty)$. Nevertheless, $\sin_p$ is real analytic on the interval $[0, \frac{\pi_p}{2})$ for all $p \in (1, \infty)$ [14].

So far we have supposed that $p, q \in (1, \infty)$, but with natural interpretations of the integrals involved the extreme values 1 and ∞ can be allowed (see [14] for more details). This gives

$$\pi_{p,q} = \begin{cases} 2p', & \text{if} \quad 1 \leq p \leq \infty, q = 1, \\ 2, & \text{if} \quad 1 \leq p \leq \infty, q = \infty, \\ \infty, & \text{if} \quad p = 1, 1 \leq q < \infty, \\ 2, & \text{if} \quad p = \infty, 1 \leq q \leq \infty. \end{cases}$$

Corresponding values of $\sin_{p,q}$ and $\cos_{p,q}$ are given by

$$\sin_{p,q}(x) = \begin{cases} 1 - (1 - x/p')^{p'}, & \text{if} \quad 1 < p \leq \infty, q = 1, \\ x, & \text{if} \quad 1 \leq p \leq \infty, q = \infty, \\ x, & \text{if} \quad p = \infty, 1 \leq q \leq \infty, \end{cases}$$

and

$$\cos_{p,q}(x) = \begin{cases} (1 - x/p')^{1/(p-1)}, & \text{if} \quad 1 < p \leq \infty, q = 1, \\ 1, & \text{if} \quad 1 \leq p \leq \infty, q = \infty, \\ 1, & \text{if} \quad p = \infty, 1 \leq q \leq \infty. \end{cases}$$

When $p = 1$ these functions can be expressed in terms of elementary functions only when q is rational [14]. Thus

$$\sin_{1,1}(x) = 1 - e^{-x}, \quad \cos_{1,1}(x) = e^{-x}, \quad \sin_{1,2}(x) = \tanh(x), \quad \cos_{1,2}(x) = (\cosh(x))^{-2}.$$

The following presents some properties of the numbers $\pi_{p,q}$ which will mainly be used in Chapters 4 and 6.

Lemma 1.1 (Monotonicity)**.** *[11, Lemma 2.1]Let $p, q \in (1, \infty)$ and let the number $\pi_{p,q}$ be defined by* (1.3)*. Then*

$$\begin{aligned} p \mapsto \pi_{p,q} \quad & \textit{is decreasing on } (1,\infty) \textit{ for any fixed } q \in (1,\infty), \\ q \mapsto \pi_{p,q} \quad & \textit{is decreasing on } (1,\infty) \textit{ for any fixed } p \in (1,\infty). \end{aligned}$$

Lemma 1.2 (Symmetry)**.** *[11, Lemma 2.2] . Let $p, q \in (1, \infty)$. Then*

$$\pi_{p,q} = \frac{p'}{q}\pi_{q',p'}.$$

Notice that for the case $p = q$,

$$\pi_p = \frac{p'}{p}\pi_{p'}.$$

Lemma 1.3. *[11, Lemma 2.3]. Let $p, q \in (1, \infty)$. Then, the following estimates hold.*

(a) If $p' \leq q$ then $\pi_{p,q} \leq \pi_{q',q}$.

(b) If $p' > q$ then $\pi_{p,q} \leq \frac{p'}{q}\pi_{p,p'}$.

Another important property is that,

$$\pi_{p',p} = |S_p|, \tag{1.9}$$

where $|S_p|$ is the area of the set enclosed by the p-circle $|x|^p + |y|^p = 1$ (see [14, Chapter 2] and [11] for more details). For $p \in [1, \infty]$, set

$$S_p = \{(x, y) \in \mathbb{R}^2; |x|^p + |y|^p \leq 1\} \text{ if } p < \infty,$$
$$S_\infty = \{(x, y) \in \mathbb{R}^2; \max(|x|, |y|) \leq 1\}.$$

Let $p_1, p_2 \in (1, \infty)$ be such that $p_1 \leq p_2$, then

$$S_1 \subset S_{p_1} \subseteq S_{p_2} \subset S_\infty,$$

and so the 2-dimensional Lebesgue measure of S_p,

$$2 = |S_1| < |S_{p_1}| \leq |S_{p_2}| < |S_\infty| = 4. \tag{1.10}$$

These estimates together with (1.9) imply the following assertion.

Lemma 1.4. *[11, Lemma 2.4] Let $p \in (1, \infty)$. Then*

$$2 \leq \pi_{p,p'} \leq 4.$$

Now, we provide some useful identities concerning (p, q)-trigonometric functions essential for the calculations in Chapters 4 and 6. Here f^{-1} denotes the inverse function of f.

Lemma 1.5. *[11, Proposition 3.2]. For all $y \in [0, 1]$,*

(a) $\cos^{-1}_{p,q}(y) = \sin^{-1}_{p,q}\left((1 - y^p)^{1/q}\right)$ *and* $\sin^{-1}_{p,q}(y) = \cos^{-1}_{p,q}\left((1 - y^q)^{1/p}\right)$.

(b) $\frac{2}{\pi_{p,q}} \sin^{-1}_{p,q}\left(y^{1/q}\right) + \frac{2}{\pi_{q',p'}} \sin^{-1}_{q',p'}\left((1 - y)^{1/p'}\right) = 1$.

(c) $(\cos_{p,q}(\pi_{p,q} y/2))^p = (\sin_{q',p'}(\pi_{q',p'}(1 - y)/2))^{p'}$.

Lemma 1.6. *[11, Proposition 3.3]. For all $p, q \in (1, \infty)$ and for all $\theta \in (0, \pi_{p,q}/2]$,*

$$\frac{2}{\pi_{p,q}} \leq \frac{\sin_{p,q}(\theta)}{\theta} \leq 1.$$

We now consider, in more detail, the particular case $p = q \in (1, \infty)$. A full account on this matter can be found in [9, Section 2] and [14, Chapter 2].

For any $p \in (1, \infty)$. According to (1.4), we have

$$\pi_p = \frac{2\pi}{p \sin(\pi/p)} \qquad \text{and} \qquad p\pi_p = 2\Gamma(1/p')\Gamma(1/p) = p'\pi_{p'}. \tag{1.11}$$

Moreover, π_p decreases as p increases (see Lemma 1.1), and is a smooth function in $p \in (1, \infty)$, such that

$$\begin{cases} \pi_p \to \infty & p \to 1^+ \\ \pi_p = \pi & p = 2 \\ \pi_p \to 2 & p \to \infty. \end{cases}$$

The following illustrates the dependence of $\sin_p(\pi_p x)$ over p.

Lemma 1.7. *[9, Corollary 4.4] Let $p_1, p_2 \in (1, \infty)$.*

(a) If $p_1 < p_2$, then

$$1 > \frac{\sin_{p_2}^{-1}(x)}{\sin_{p_1}^{-1}(x)} \geq \frac{\pi_{p_2}}{\pi_{p_1}}, \quad x \in (0, 1].$$

(b) If $p_1 \leq p_2$, then

$$\sin_{p_1}^{-1}(x) \geq \sin_{p_2}^{-1}(x) \quad \text{and} \quad \frac{1}{\pi_{p_2}} \sin_{p_2}^{-1}(x) \geq \frac{1}{\pi_{p_1}} \sin_{p_1}^{-1}(x), \quad x \in [0, 1].$$

(c) If $p_1 \leq p_2$, then

$$\sin_{p_1}(\pi_{p_1} x) \geq \sin_{p_2}(\pi_{p_2} x), \quad x \in [0, 1/2].$$

Now we recall various elementary properties of the p-trigonometric functions and their connections with classical analysis.

A natural change of variable to $z = x^{-1}t$ and then to $u = z^p$ will yield

$$\begin{aligned}\sin_p^{-1}(x) &= \int_0^x (1-t^p)^{-1/p}\mathrm{d}t \\ &= x\int_0^1 (1-x^p z^p)^{-1/p}\mathrm{d}z \\ &= \frac{x}{p}\int_0^1 u^{1/p-1}(1-x^p u)^{-1/p}\mathrm{d}u. \end{aligned} \tag{1.12}$$

Hence the representation

$$\sin_p^{-1}(x) = xF\left(\frac{1}{p}, \frac{1}{p}; 1+\frac{1}{p}; x^p\right), \qquad \forall x \in [0,1), \tag{1.13}$$

where F on the right side denotes the confluent hypergeometric function [9]. This leads naturally to an expression in terms of the incomplete beta function $I(a, b; .)$ defined for any positive a and b by,

$$I(a,b;x) := \frac{1}{B(a,b)}\int_0^x t^{a-1}(1-t)^{b-1}\mathrm{d}t, \qquad \forall x \in [0,1],$$

[18, 8.391 & 8.392, pp. 910] as we shall see next.

For any $p > 1$, define

$$\begin{aligned}I_p(x) &:= I\left(\frac{1}{p}, \frac{1}{p'}; x^p\right) \\ &= \frac{1}{B(1/p, 1/p')}\int_0^{x^p} t^{1/p-1}(1-t)^{-1/p}\mathrm{d}t.\end{aligned}$$

Then, according to (1.11) and by changing the variable to $u = x^{-p}t$ we get

$$I_p(x) = \frac{2}{p\pi_p}x\int_0^1 u^{1/p-1}(1-x^p u)^{-1/p}\mathrm{d}u. \tag{1.14}$$

Statement (1.12) together with (1.14) yield

$$I_p(x) = \frac{2}{\pi_p}\sin_p^{-1}(x), \qquad \forall x \in [0,1]. \tag{1.15}$$

1.2 Contribution

The main aim of this thesis is to answer the following question: Consider a continuous 2-periodic function $f : \mathbb{R} \longrightarrow \mathbb{C}$ possibly given in terms of parameters (e.g. the parameters are $p, q \in (1, \infty)$ when $f_n(\cdot) = \sin_{p,q}(n\cdot)$). Denote by $\mathfrak{E}_f$ the family of dilations $\mathfrak{E}_f := \{f_n\}_n$, such that $f_n(\cdot) := f(n\cdot)$ for all $n \in \mathbb{N}$. When does $\mathfrak{E}_f$ form a basis of $L^r(0, 1)$ for $r \in (1, \infty)$? This problem can be phrased as a question about the values of the parameters p, q such that the family $\mathfrak{E}_f$ generates a basis in $L^r(0, 1)$ for $r \in (1, \infty)$.

We address this question in several different settings in Chapters 3-7. All these chapters contain new results which extend in various ways those of the existing literature (see Tables 1.1 and 1.2 on page 12). We shall briefly describe what these contributions are.

A general result is presented in Chapter 3 where we introduce an improved *one-term* (Corollary 3.1) and new *multi-term* criteria (Corollaries 3.2 and 3.3) for determining Schauder and Riesz bases properties of the family $\mathfrak{E}_f$, respectively, in the context of Lebesgue spaces. These criteria provide conditions sufficient for the invertibility of the change of coordinates map T between a basis $\{e_n\}_n$ of $L^r(0, 1)$ and the family $\mathfrak{E}_f$. Our starting point is the criteria for Riesz basis in the case $r = 2$ formulated in Lemmas 2.5, 2.6 and 2.7. Then we generalise this by re-examining the approach of several complex variables established in [32], in the context of [7]. The main device T is a linear operator of the Banach space which is defined as a linear combination of certain isometries. We begin the chapter by presenting the so called *one-term* criterion originally developed in [5], and which attracted a series of research studies concerning bases properties of generalised trigonometric functions. Then we illustrate a detailed configuration of the new criteria whose key point is the association of a partial sum of the T-expansion with a polynomial in several complex variables. This mechanism enables the study of the invertibility of T in terms of the analytic properties of the polynomial. The criterion introduces conditions sufficient for the invertibility of T which involve localising the zeros and minimising the modulus of the associated polynomial. In Corollaries 3.1 and 3.2, we introduce an approach combined with the criteria above, which improves the thresholds of invertibility of T in general. The approach was proved fruitful when applied to $\mathfrak{E}_{\sin_p} := \{\sin_p(n\pi_p\cdot)\}_{n\in\mathbb{N}}$ and $\mathfrak{E}_{\cos_p} := \{\cos(n\pi_p.)\}_{n\in\mathbb{N}_0}$ in Chapter 4.

One of the main applications of the results established in Chapter 3 is to investigate the bases properties of the family $\mathfrak{E}_f$ when f is a generalised trigonometric function given in terms of the parameters $(p,q) \in (1,\infty)^2$ for $p \neq q$ and $p = q$.

Chapter 4 examines bases and regularity properties of $\mathfrak{E}_{\cos_p}$ and $\mathfrak{E}_{\sin_p}$ using the *one-term* (Theorem 3.1) and the improved *one-term* (Corollary 3.1) criteria. Sharp estimates for the Fourier coefficients of $\cos_p$ were found in two cases $p \in (1,2)$ and $p \in (2,\infty)$. Two thresholds $p_0 \in (1,2)$ and $p_1 \in (2,\infty)$ were obtained via some delicate analytical computations such that the family $\mathfrak{E}_{\cos_p}$ forms a basis of $L^r(0,1)$ for all $p \in [p_0,p_1]$ and $r \in (1,\infty)$. Applying Corollary 3.1 by means of some numerical approximations to both sets of functions allowed further extensions in the p-thresholds. The families $\mathfrak{E}_{\cos_p}$ and $\mathfrak{E}_{\sin_p}$ are Schauder bases for all $p \in (\hat{p}_{0,1}, \hat{p}_{1,1})$ and $p \in (\tilde{p}_{1,1}, \infty)$ respectively, such that $\hat{p}_{0,1} < p_0$, $\hat{p}_{1,1} > p_1$ and $\tilde{p}_{1,1} < \tilde{p}_1$. These results provide improvements upon those of [12] and [7] (see Table 1.1).

Chapter 5 illustrates the rich structure behind the *multi-term* criteria in the case $r = 2$ (Riesz basis) when applied to the p-sine, the p-cosine and the p-exponential functions. Both Corollaries 3.2 and 3.3 show the possibility of further progress in the p-thresholds obtained, improving those of Chapter 4 when $f = \cos_p$ and those of [7] when $f = \sin_p$ (see Table 1.1 for more details).

Chapter 6 investigates bases and regularity properties of the generalised trigonometric functions $\sin_{p,q}$ and $\cos_{p,q}$. Monotonicity properties of the $\sin_{p,q}$ and the $\cos_{p,q}$ functions are achieved. Upper bounds for the Fourier coefficients of these functions are given. Conditions are obtained under which the family $\mathfrak{E}_{\cos_{p,q}} := \{\cos_{p,q}(n\pi_{p,q}\cdot)\}_{n\in\mathbb{N}_0}$ generates a basis of every Lebesgue space $L^r(0,1)$ with $r \in (1,\infty)$; when q is the conjugate of p it is sufficient to require that $p \in [\mathbf{p}_1, \mathbf{p}_2]$, where $\mathbf{p}_1 < 2$ and $\mathbf{p}_2 > 2$ are computable numbers. These results sharpen previously known ones in [12] as illustrated in Table 1.2. In addition, a comparison is made of the speed of decay of the Fourier sine coefficients of a function in Lebesgue and

Lorentz sequence spaces with that of the corresponding coefficients with respect to the functions $\sin_{p,q}$. Theorem 6.1 illustrates the possibility of further progress outside the interval obtained in [13] in the setting of Lebesgue spaces. We also show (Remark 6.1) that results can be achieved in the context of Lorentz sequence spaces that involve loss of sharpness of the exponents. Both studies are investigated under the assumption of monotonicity of the sequences $(|\mathbf{a}_k|)_k$ and $(|\mathbf{b}_k|)_k$.

The last contribution of the thesis is to answer the first question but now in the setting of Hardy spaces $H^r(\mathbb{T}^\infty)$ for $r \in (1, \infty)$. Let $\{\mathfrak{h}_n\}_n$ be the sequence of monomials in $H^r(\mathbb{T}^\infty)$. For any $\varphi \in H^r(\mathbb{T}^\infty)$, we aim at investigating the conditions under which the family $\mathfrak{H}_\varphi := \{\varphi\mathfrak{h}_n\}_{n\in\mathbb{N}}$ is a basis in $H^r(\mathbb{T}^\infty)$. The key element is the case $r = 2$. Using Parseval's identity we confirm the existence of an invertible isometry U mapping $L^2(0, 1)$ into $H^2(\mathbb{T}^\infty)$. Applying the operator T of Chapter 3 to a given element g in $L^2(0, 1)$ will result in multiplying (point-wise) φ by the U-transformed side of g. This concept was utilised in the new *multi-term* Riesz basis criteria in Chapter 3. Chapter 7 introduces some fundamental properties of the multipliers on Hardy spaces $H^r(G)$ and calculates their norms for $G = \mathbb{D}^n, \mathbb{T}^n$ and $\mathbb{T}^\infty$. It also establishes new criteria concerning the study of bases properties of the families $\mathfrak{H}_\varphi$. These criteria are formulated based on the analyticity properties of the multipliers φ and the localisation of their zeros with respect to $\overline{\mathbb{D}}^\infty$.

Chapters 3 and 7 of this thesis give rise to a fascinating conjecture which we now outline. Is there a mechanism which allows the study of bases properties of the family $\mathfrak{E}_f$ of Banach spaces $L^r(0, 1)$ in terms of the corresponding sequence in the Hardy spaces $H^r(\mathbb{T}^\infty)$, $r \neq 2$? The lack of Parseval's identity in the non-Hilbertian case leads to seek other type of methods to handle this question.

Below we summarise these contributions by demonstrating, in a chronological order, the improvements we have had in the (p, q)-thresholds for the bases properties of the families considered in the Lebesgue spaces $L^r(0, 1)$ for $r \in (1, \infty)$.

p-**thresholds**	p-**cosine**	p-**sine**
One-term Schauder basis (see [12] and [7])	$(p_0^\dagger, 2], p_0^\dagger \approx 1.75$	$[\tilde{p}_1, \infty), \tilde{p}_1 \approx 1.087$
One-term Schauder basis (Chapter 4)	$[p_0, p_1]$, $p_0 \approx 1.458801, p_1 \approx 2.42865$	
Improved one-term Schauder basis (Chapter 4)	$(\hat{p}_{0,1}, p_0) \cup (p_1, \hat{p}_{1,1}), \hat{p}_{0,1} \approx 1.2978,\ \hat{p}_{1,1} \approx 3.2205$ for $k = 201$	$(\tilde{p}_{1,1}, \infty), \tilde{p}_{1,1} \approx 1.0484$ for $k = 141$
Multi-term Riesz basis (Chapter 5)	$[p_{0,1}, p_0] \cup [p_1, p_{1,1}],\ p_{0,1} \approx 1.441908, p_{1,1} \approx 2.462328$	
Multi-term Riesz basis (Chapter 5)	$[p_{0,2}, p_{0,1}] \cup [p_{1,1}, p_{1,2}], p_{0,2} \approx 1.400566, p_{1,2} \approx 2.561986$	
Improved multi-term Riesz basis (Chapter 5)	$[p_{0,3}, p_{0,2}] \cup [p_{1,2}, p_{1,3}], p_{0,3} \approx 1.296718, p_{1,3} \approx 3.339563$ for $k = 321$	$(q_0, \infty), q_0 \approx 1.038537$ for $k = 201$

Table 1.1: The best thresholds achieved using *one-term* and improved *one-term* Schauder basis criteria, and also *multi-term* and improved *multi-term* Riesz basis criteria such that the families $\mathfrak{E}_{\cos_p}$ and $\mathfrak{E}_{\sin_p}$ form bases of $L^r(0,1), r \in (1,\infty)$.

p-**thresholds**	(p,p')-**cosine**	(p,p')-**sine**
One-term Schauder basis (see [12] and [11])	$(\mathbf{p}_0, 2), \mathbf{p}_0 \approx 1.8$	$p \in (1,\infty)$
One-term Schauder basis (Chapter 6)	$[\mathbf{p}_1, \mathbf{p}_2], \mathbf{p_1} \approx 1.487807, \mathbf{p}_2 \approx 2.526402$	

Table 1.2: The best thresholds achieved using *one-term* Schauder basis criterion such that the families $\mathfrak{E}_{\cos_{p,p'}}$ and $\mathfrak{E}_{\sin_{p,p'}}$ form Schauder bases of $L^r(0,1), r \in (1,\infty)$.

Chapter 2

Background

2.1 Bases of Banach and Hilbert spaces

Let X be a Banach space with norm $\|\cdot\|$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X is called a (*Schauder*) basis of X if, for any $x \in X$, there exists a unique sequence of scalars $(c_n)_{n\in\mathbb{N}}$, dependent continuously on x, such that

$$x = \sum_{n=1}^{\infty} c_n x_n; \text{ that is, } \lim_{N\to\infty} \left\| \sum_{n=1}^{N} c_n x_n - x \right\|_X = 0.$$

If X is a Hilbert space, a basis $\{x_n\}_{n\in\mathbb{N}}$ of X is called a *Riesz basis* if the map $(c_n)_n \mapsto \sum_{n=1}^{\infty} c_n x_n$ is an isomorphism of l_2 onto X. This means that, there are two positive constants c and C such that for all $(c_n)_n \in l_2$,

$$c \sum_{n=1}^{\infty} |c_n|^2 \leq \left\| \sum_{n=1}^{\infty} c_n x_n \right\|_2^2 \leq C \sum_{n=1}^{\infty} |c_n|^2.$$

It is plain that any complete orthonormal system in a separable Hilbert space is a Riesz basis. Examples of such systems are the sequence of trigonometric functions $(e^{in\pi x})_{n\in\mathbb{Z}}$ in $L^2(-1,1)$ and the sequence of standard unit vectors in l_2.

Outside the world of Hilbert spaces with its strong geometrical flavour provided by the notion of orthogonality, more effort is often needed to produce examples of bases. For example, when $r \in (1,\infty)$, a basis of $L^r(-1,1)$ is given by $(e^{in\pi x})_{n\in\mathbb{Z}}$. This follows from a result due to M. Riesz.

Lemma 2.1 (M. Riesz). *[15, Chapter 12, Section 10, pp. 106] For $r \in (1, \infty)$, a basis of $L^r(-1,1)$ is given by $(e^{in\pi x})_{n\in\mathbb{Z}}$, i.e.*

$$\lim_{N\to\infty} \left\| f - \sum_{|n|\le N} c_n e^{in\pi x} \right\|_r = 0,$$

for all $f \in L^r(-1,1)$, where $c_n = \frac{1}{2}\int_{-1}^{1} f(x)e^{-in\pi x}\mathrm{d}x$ and $\|\cdot\|_r$ is the L^r-norm; when $r = 1$ the limit is false.

Consequently, given any $f \in L^r(0,1)$, its odd extension to $L^r(-1,1)$ has a unique representation in terms of the $\sin(n\pi x)$, which means that $\mathfrak{E}_{\sin} := \{\sin(n\pi\cdot)\}_{n\in\mathbb{N}}$ is a basis of $L^r(0,1)$. A similar argument applies to $\mathfrak{E}_{\cos} := \{\cos(n\pi x)\}_{n\in\mathbb{N}_0}$ via an even extension.

Lemma 2.2 (Riemann-Lebesgue Lemma). *[19, Proposition 2.2.17] Let $f \in L^1(-1,1)$. Then its n-th Fourier coefficient c_n tends to zero as $n \to \infty$. That is,*

$$c_n = \frac{1}{2}\int_{-1}^{1} f(x)e^{-in\pi x}\mathrm{d}x \longrightarrow 0 \qquad \textit{as} \qquad n \to \infty.$$

Theorem 2.1 (Carleson-Hunt). *[17] For every $r \in (1,\infty)$. The Fourier series of an L^r function on $(-1,1)$ converges almost everywhere. That is, for $f \in L^r(-1,1)$*

$$\lim_{N\to\infty} \sum_{|n|\le N} c_n e^{in\pi x} = f(x) \quad \textit{a.e on} \quad (-1,1).$$

Definition 2.1. *[21] We call two sequences $(u_n)_n$ and $(v_n)_n$ in the Banach space X equivalent, if there exists a bounded linear bijective operator T on X such that $Tu_n = v_n$ for every $n \in \mathbb{N}$.*

Theorem 2.2. *[21] Let $(u_n)_n$ and $(v_n)_n$ be equivalent sequences in a Banach space X. Then $(u_n)_n$ is a Schauder basis if, and only if, $(v_n)_n$ is a Schauder basis.*

Proof. Since $(u_n)_n$ and $(v_n)_n$ are equivalent, then there exists a bounded bijective operator T such that $Tu_n = v_n$, $\forall n \in \mathbb{N}$.

Suppose that $(u_n)_n$ is a Schauder basis in X, then there exists a unique sequence $(c_n)_n$ dependent continuously on x such that $x = \sum_{n=1}^{\infty} c_n u_n$.

For any $y \in X$ there exists a unique $x \in X$ such that $y = Tx$. Moreover,

$$\begin{aligned}\lim_{N\to\infty}\left\| y - \sum_{n=1}^{N} c_n v_n \right\|_X &= \lim_{N\to\infty}\left\| y - \sum_{n=1}^{N} c_n T u_n \right\|_X \\ &= \lim_{N\to\infty}\left\| Tx - T\left(\sum_{n=1}^{N} c_n u_n\right) \right\|_X \\ &\leq \|T\| \lim_{N\to\infty}\left\| x - \sum_{n=1}^{N} c_n u_n \right\|_X = 0.\end{aligned}$$

The result follows.

The proof of the other implication follows similarly by using T^{-1} instead. $\square$

2.1.1 Bases properties of (p, q)-trigonometric functions

In recent years there has been a remarkable interest in the so-called (p, q)-trigonometric functions and their bases properties. This section highlights some of these results briefly, however for further details we recommend the following references [5], [14], [9], [12] and [8] and Tables 1.1 and 1.2.

Let $p, q \in (1, \infty)$ and $f_n(\cdot) = \sin_{p,q}(n\pi_{p,q}\cdot)$. Since each of these functions is continuous on $[0, 1]$ and has an odd extension to $\mathbb{R}$, the Fourier sine expansion has the form

$$\sin_{p,q}(n\pi_{p,q}x) = \sum_{j=1}^{\infty} \widehat{f}_n(j)\sin(j\pi x), \quad \widehat{f}_n(j) = 2\int_0^1 \sin_{p,q}(n\pi_{p,q}x)\sin(j\pi x)\mathrm{d}x.$$

Let $\tau_j(p, q) := \widehat{f}_1(j)$ be the Fourier coefficients of the first element of the sequence $\mathfrak{E}_{\sin_{p,q}}$, that is $\sin_{p,q}(\pi_{p,q}x)$.

Lemma 2.3. *[7, Lemma 4.2] For $j = 3$ or $j = 9$, we have $\tau_j(p, q) < \tau_1(p, q)$ for any $p, q \in (1, \infty)$.*

Since $\sin_{p,q}(\pi_{p,q}x)$ is symmetric around $x = \frac{1}{2}$, $\tau_j(p, q) = 0$ for every even and positive integer j. Moreover, for $n > 1$

$$\widehat{f}_n(k) = \begin{cases} \tau_j(p, q) & \text{for } jn = k, \\ 0 & \text{otherwise.} \end{cases}$$

Define the operator $T : L^r(0,1) \longrightarrow L^r(0,1)$ by means of

$$T = \sum_{j=1}^{\infty} \tau_j(p,q) M_j, \tag{2.1}$$

where $M_j(\sin(n\pi\cdot)) = \sin(jn\pi\cdot)$ for $j, n \in \mathbb{N}$ are isometries mapping $L^r(0,1)$ into itself.

The *one-term* criterion stated in our Theorem 3.1 below will now take the following form. If

$$\sum_{j=3}^{\infty} |\tau_j(p,q)| < |\tau_1(p,q)|, \tag{2.2}$$

then $\mathfrak{E}_{\sin_{p,q}}$ forms a Schauder basis of $L^r(0,1)$ for all $r \in (1,\infty)$. It is in trying to satisfy this inequality that the restriction on (p,q) appears.

The same technique can also be applied to the set $\mathfrak{E}_{\cos_{p,q}}$ for which an even extension to $\mathbb{R}$ is required (see Remark 3.2 and Section 4.3).

Bases properties of the generalised trigonometric functions were first examined in [5], where it was shown that the family $\mathfrak{E}_{\sin_p}$ forms a Schauder basis of $L^r(0,1)$ for all $r \in (1,\infty)$ and $p \geq \frac{12}{11}$. Further development in this respect were settled in [9], [11] and [7]. Currently we know that this family is a Schauder basis of every Lebesgue space $L^r(0,1)$, $r \in (1,\infty)$ provided that $p \in [\tilde{p}_1, \infty)$ and also a Riesz basis of $L^2(0,1)$ for $p \in (\hat{p}_1, \tilde{p}_1]$, where $\tilde{p}_1 \approx 1.087$ and $\hat{p}_1 \approx 1.044$ satisfy complicated equations involving hypergeometric functions [7]. Moreover it was shown in [11] that this basis property is also fulfilled by the set $\mathfrak{E}_{\sin_{p,q}}$ provided that $\frac{p'}{q} < \frac{4}{\pi^2-8}$. In particular, this shows that for all $p \in (1,\infty)$, the family $\mathfrak{E}_{\sin_{p,p'}}$ forms a basis of every $L^r(0,1)$ for all $p, r \in (1,\infty)$ (we recommend the reader to check the Tables 1.1 and 1.2 for further details). These bases properties mean that given any $r \in (1,\infty)$ and $f \in L^r(0,1)$, then for appropriate p and q, there exist unique sequences of real numbers $(\mathbf{a}_k)_k$ and $(\mathbf{b}_k)_k$ such that

$$f(x) = \sum_{k=1}^{\infty} \mathbf{a}_k \sin(k\pi x) = \sum_{k=1}^{\infty} \mathbf{b}_k \sin_{p,q}(k\pi_{p,q} x).$$

From the various results established in the recent paper [12], it follows that $\mathfrak{E}_{\cos_p}$

is a Schauder basis of $L^r(0,1)$ for all $r \in (1,\infty)$ and $p \in (p_0^\dagger, 2]$ where $p_0^\dagger \approx 1.75$. It was also shown that the functions $C_{1/p}(n\pi_{p,p'}\cdot)(n \in \mathbb{N}_0)$ defined by (1.1), form a basis of $L^r(0,1)$ for all $r \in (1,\infty)$ whenever $p \in (1,\infty)$; and so do the family $\mathfrak{E}_{\cos_{p,p'}}$ provided that $p \in (\mathbf{p}_0, 2)$ such that

$$\frac{\pi_{p,p'}^2}{p-1} = \frac{8\pi}{\pi^2 - 8} \qquad \text{at} \qquad p = \mathbf{p}_0, \tag{2.3}$$

and $\mathbf{p}_0$ lies below 1.8.

According to [9], criterion (2.2) ceases to hold true for $p \approx 1.04399$ for the family $\mathfrak{E}_{\sin_p}$, therefore it was essential to tackle the basisness question by different means in the regime $p \to 1$. In [7, Section 4], besides the improvements mentioned above, the authors found two further criteria for the invertibility of T which generalise (2.2) in the Hilbert space setting $r = 2$. The test is amenable to analytical and numerical investigations and involves finding sharp bounds on the first few coefficients $\tau_j(p,q)$.

Lemma 2.4. *[7, Theorem 6.5] The family $\mathfrak{E}_{\sin_p}$ is a Schauder basis of $L^r(0,1)$ for $p \in (\tilde{p}_1, \frac{6}{5})$, where $\tilde{p}_1$ is the solution of*

$$\pi_p = \frac{[a_1(p) - a_3(p) - a_5(p) - a_7(p)]\pi^2}{4(\pi^2/8 - 1 - 1/9 - 1/25 - 1/49)},$$

and is numerically equal to 1.087063.

The main idea behind the *multi-term* criteria to be discussed in Chapter 3 is as follows. Consider the three Fourier coefficients $\tau_j(p,q), (j = 1,3,9)$.

Lemma 2.5. *[7, Corollary 4.3] Let $p, q \in (1,\infty)$ be such that $\tau_9(p,q) \pm |\tau_3(p,q)| + \tau_1(p,q) > 0$ and $|\tau_3(p,q)\,(\tau_1(p,q) + \tau_9(p,q))\,| \geq |4\tau_9(p,q)\tau_1(p,q)|$.*
If

$$\sum_{j \notin \{1,9\}}^{\infty} |\tau_j(p,q)| < \tau_1(p,q) + \tau_9(p,q),$$

then the family $\mathfrak{E}_{\sin_{p,q}}$ forms a Riesz basis of $L^2(0,1)$.

A second generalisation claims

Lemma 2.6. *[7, Corollary 4.4] Let $p, q \in (1,\infty)$ be such that $\tau_9(p,q) \pm |\tau_3(p,q)| + \tau_1(p,q) > 0, \tau_9(p,q) > 0$ and $|\tau_3(p,q)(\tau_1(p,q) + \tau_9(p,q))| < |4\tau_9(p,q)\tau_1(p,q)|$.*

If

$$\sum_{j\notin\{1,3,9\}}^{\infty} |\tau_j(p,q)| < (\tau_1(p,q)-\tau_9(p,q))\left(1-\frac{\tau_3^2(p,q)}{4\tau_1(p,q)\tau_9(p,q)}\right)^{\frac{1}{2}},$$

then the family $\mathfrak{E}_{\sin_{p,q}}$ *forms a Riesz basis of* $L^2(0,1)$.

The approach employed in the proof of Lemma 2.4 combined with the criteria of Lemma 2.5 yield

Lemma 2.7. *[7, Proposition 7.1] Let* $r=2$ *and* $5\le k\not\equiv_2 0$. *Let* $p,q\in(1,\infty)$ *be such that*

(a) $\tau_3(p,q)>0, \tau_9(p,q)>0$ *and* $\tau_j(p,q)\ge 0$ *for all other* $5\le j\le k$.

(b) $\tau_9(p,q)\pm\tau_3(p,q)+\tau_1(p,q)>0$.

(c) $\tau_3(p,q)(\tau_1(p,q)+\tau_9(p,q))>4\tau_9(p,q)\tau_1(p,q)$.

If

$$\pi_{p,q} < \left(\tau_1(p,q)+\tau_9(p,q)-\sum_{\substack{3\le j\le k\\ j\notin\{1,9\}}}\tau_j(p,q)\right)\frac{\pi^2}{4(\pi^2/8-\sum_{\substack{j=1\\ j\equiv_2 1}}^{k}(1/j^2))},$$

then the family $\mathfrak{E}_{\sin_{p,q}}$ *is a Riesz basis of* $L^2(0,1)$.

Recently, it was shown in [13] that the inverse of the map T defined above has properties similar to those of T.

Lemma 2.8. *[13, Lemma 2.1] Let* $p,q,r\in(1,\infty)$, *let* T *be as in* (2.1) *and suppose that* (2.2) *holds. Then, there are constants* $\beta_{2k+1}:=\beta_{2k+1}(p,q)$ *for* $k\in\mathbb{N}$ *with*

$$\sum_{k=1}^{\infty}|\beta_{2k+1}|<\infty,$$

such that $T^{-1}: L^r(0,1)\longrightarrow L^r(0,1)$ *has the representation*

$$T^{-1}=\frac{1}{\tau_1}\,\mathrm{Id}+\sum_{k=1}^{\infty}\beta_{2k+1}M_{2k+1},$$

where Id *is the identity operator of* $L^r(0,1)$ *to itself and* M_{2k+1} *are the isometries in* (2.1).

The same paper also established a limited rearrangement property of the basis elements in $L^r(0,1)$, $r \in (1,\infty)$.

Lemma 2.9. *[13, Lemma 2.2] Let $p, q, r \in (1,\infty)$, let T be as in (2.1) and suppose that (2.2) holds. Let $f \in L^r(0,1)$ be given in terms of the bases $\mathfrak{E}_{\sin}$ and $\mathfrak{E}_{\sin_{p,q}}$ by*

$$f(x) = \sum_{k=1}^{\infty} \mathbf{a}_k \sin(k\pi x) = \sum_{k=1}^{\infty} \mathbf{b}_k \sin_{p,q}(k\pi_{p,q} x), \quad x \in (0,1). \tag{2.4}$$

Then,

$$\mathbf{a}_k = \sum_{\substack{m,n\in\mathbb{N} \\ mn=k}} \mathbf{b}_n \tau_m(p,q), \qquad \forall k \in \mathbb{N}. \tag{2.5}$$

This was essentially used in proving the following statement, illustrated in Lemma 2.10, which investigates the relationship between the decay properties of the Fourier sine coefficients of a function and those of the corresponding coefficients when the classical sine functions are replaced by the $\sin_{p,q}$ functions.

Lemma 2.10. *[13, Theorem 2.4] Let $p, q, r \in (1,\infty)$ be such that (2.2) holds and let $f \in L^r(0,1)$ have the representation (2.4). Suppose that the sequences $(|\mathbf{a}_k|)_{k\in\mathbb{N}}$ and $(|\mathbf{b}_k|)_{k\in\mathbb{N}}$ are non-increasing and let $\alpha \in (0,2)$. Then, $|\mathbf{b}_k| \lesssim k^{-\alpha}$ if and only if $|\mathbf{a}_k| \lesssim k^{-\alpha}$.*

Next we highlight some of the important definitions and properties of spaces of analytic functions which are going to be useful for the study of the *multi-term* case in Chapters 3 and 7.

2.2 Hardy spaces on polydiscs

Throughout this chapter and the following ones, $\mathbb{C}$ will denote the complex field, $\mathbb{C}^n$ will be the Cartesian product of n copies of $\mathbb{C}$, while $\mathbb{C}^\infty$ will denote the Cartesian product of infinitely many copies of $\mathbb{C}$. The points of $\mathbb{C}^n$ are thus ordered n-tuples $z = (z_1, \dots, z_n)$, with each $z_i \in \mathbb{C}$. Algebraically, it is an n-dimensional vector space over the field $\mathbb{C}$; topologically, $\mathbb{C}^n$ is the Euclidean space of dimension $2n$.

Definition 2.2. *[35, Appendix B7., pp. 254] If $\{G_\alpha\}$ is a collection of abelian groups, their complete direct sum is the group G defined as follows: G, as a set, is the Cartesian product of the sets G_α, and addition is performed coordinate-wise: If $x, y \in G$, $x + y \in G$ such that the α-th coordinate is $x(\alpha) + y(\alpha)$.*

The direct sum of the groups G_α is the subgroup of their complete direct sum which consists of all x which have $x(\alpha) \neq 0$ for only finitely many α.

We recall Tychonoff's theorem.

Theorem 2.3 (Tychonoff)**.** *[35] The direct sum of any finite collection of locally compact Abelian groups is a locally compact Abelian group. The complete direct sum of any collection of compact Abelian groups is a compact Abelian group.*

2.2.1 Finite and infinite dimensional torus and polydisc

The open unit disc in $\mathbb{C}$ is denoted by $\mathbb{D}$; its boundary is the circle $\mathbb{T}$. The unit polydisc $\mathbb{D}^n$ and the torus $\mathbb{T}^n$ are the subsets of $\mathbb{C}^n$ which are complete direct sums of n copies of $\mathbb{D}$ and $\mathbb{T}$, respectively. Thus

$$\mathbb{D}^n = \{(z_1, \dots, z_n) : |z_j| < 1 \ \forall j = 1, \dots, n\},$$

which topologically has dimension $2n$ in the Euclidean $2n$-space, while the boundary Γ^n is of dimension $2n - 1$ and is defined as the union of the Cartesian products of the circumferences $|z_j| = 1$ and the discs $|z_k| \leq 1$ for $k \neq j$ and $k, j = 1, \dots, n$. The torus

$$\mathbb{T}^n = \{(z_1, \dots, z_s) : |z_j| = 1 \ \forall j = 1, \dots, n\}$$

constitutes only a small part of Γ^n when $n > 1$ with a dimension n in the $2n$-dimensional Euclidean space. But it is the part that matters the most[1] and is usually called the distinguished boundary of $\mathbb{D}^n$. $\mathbb{T}^n$ is a compact Abelian group (with component-wise multiplication as group operation) and is equipped with a Haar measure $\mathrm{d}\sigma_n = \frac{\mathrm{d}\theta_1}{2\pi} \cdots \frac{\mathrm{d}\theta_n}{2\pi}$, where $\frac{\mathrm{d}\theta_j}{2\pi}$ is the ordinary Lebesgue measure on the j-th copy of $\mathbb{T}$ divided by 2π so that $\sigma_n(\mathbb{T}^n) = 1$.

[1] See Theorem 2.6 below.

On the other hand, $\mathbb{T}^\infty$ and $\mathbb{D}^\infty$ are subsets of $\mathbb{C}^\infty$ and are defined as complete direct sums of infinitely many copies of $\mathbb{T}$ and $\mathbb{D}$, respectively. $\mathbb{T}^\infty$ is a locally compact Abelian group (see, Theorem 2.3 by Tychonoff) which is endowed by a Haar measure $\mathrm{d}\sigma = \frac{\mathrm{d}\theta_1}{2\pi}\frac{\mathrm{d}\theta_2}{2\pi}\cdots$ with $\sigma(\mathbb{T}^\infty) = 1$ and is known as the distinguished boundary of $\mathbb{D}^\infty$.

2.2.2 Dual groups and Fourier analysis

Definition 2.3 (Characters)**.** *[35, Section 1.2.1] A complex function $\mathcal{X}$ on a locally compact Abelian group G is called a character of G if $|\mathcal{X}(x)| = 1$ for all $x \in G$ and if the functional equation $\mathcal{X}(x+y) = \mathcal{X}(x)\mathcal{X}(y)$ for $x, y \in G$ is satisfied. The set of all continuous characters of G forms a group $\tilde{G}$, the dual group of G, if addition is defined by*

$$(\mathcal{X}_1 + \mathcal{X}_2)(x) = \mathcal{X}_1(x)\mathcal{X}_2(x), \qquad x \in G;\ \mathcal{X}_1, \mathcal{X}_2 \in \tilde{G}.$$

Theorem 2.4. *[35, Theorem 1.2.5] If G is discrete, $\tilde{G}$ is compact. If G is compact, $\tilde{G}$ is discrete.*

Theorem 2.5. *[35, Theorem 2.2.3] If G is the complete direct sum of a family $\{G_\alpha\}$ of compact Abelian groups, then $\tilde{G}$ is the direct sum of the corresponding dual groups $\tilde{G}_\alpha$.*

Examples 2.1. *The "classical groups" of Fourier analysis are:*

(a) $G = \mathbb{T}$: The additive group of the reals modulo 2π, or, equivalently, the circle group $\mathbb{T}$, the multiplicative group of all complex numbers of absolute values equal to 1*. The characters of $\mathbb{T}$ are of the form*

$$\mathcal{X}(\theta) = e^{in\theta}$$

and they satisfy $\mathcal{X}(\theta + 2\pi) = \mathcal{X}(\theta)$ for $\theta \in G$. Hence n must be integer, and $\tilde{G}$ is identified as the discrete group $\mathbb{Z}$. In which case the Fourier transform has the following form:

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}\mathrm{d}\theta, \qquad n \in \mathbb{Z}.$$

(b) $G = \mathbb{Z}$ *: The additive group* $\mathbb{Z}$ *of the integers. The characters of* $\mathbb{Z}$ *are of the form*

$$\mathcal{X}(n) = e^{in\theta}.$$

The correspondence $\mathcal{X} \leftrightarrow e^{i\theta}$ *is an isomorphism between* $\tilde{G}$ *and* $\mathbb{T}$*, and we conclude that* $\mathbb{T}$ *is the dual group of* $\mathbb{Z}$ *(the two topologies coincide). In this case the Fourier transform has the form:*

$$\widehat{f}(e^{i\theta}) = \sum_{n=-\infty}^{\infty} f(n)e^{-in\theta}, \qquad e^{i\theta} \in \mathbb{T}.$$

(c) $\mathbb{T}^n$ *and* $\mathbb{Z}^n$ *are the duals of each other. The Fourier coefficients* $\widehat{f}(\nu)$ *of a function* $f \in L^1(\mathbb{T}^n)$ *are defined as*

$$\widehat{f}(\nu) = \int_{\mathbb{T}^n} f(e^{i\theta_1}, \ldots, e^{i\theta_n})e^{-i\sum_{j=1}^{n}\nu_j\theta_j}\frac{\mathrm{d}\theta_1}{2\pi}\cdots\frac{\mathrm{d}\theta_n}{2\pi}, \qquad \forall \nu \in \mathbb{Z}^n.$$

(d) The infinite dimensional torus $G = \mathbb{T}^\infty$ *has a dual group* $\tilde{G} = \mathbb{Z}_0^\infty$ *which, equivalently, can also be defined as the direct sum of countably many copies of* $\mathbb{Z}$*. Functions on* $\mathbb{T}^\infty$ *can be regarded as periodic functions in countably many variables. If* $f \in L^1(\mathbb{T}^\infty)$*, then*

$$\widehat{f}(\nu) = \int_{\mathbb{T}^\infty} f(e^{i\theta_1}, e^{i\theta_2}, \ldots)e^{-i\sum_{j=1}^{\infty}\nu_j\theta_j}\mathrm{d}\sigma, \qquad \forall \nu \in \mathbb{Z}_0^\infty, \tag{2.6}$$

where only finitely many of the integers ν_j *are different from* 0*, and the* θ_j *are real numbers modulo* 2π*. The inversion formula has the form*

$$f(e^{i\theta_1}, e^{i\theta_2}, \ldots) = \sum_{\nu \in \mathbb{Z}_0^\infty} \widehat{f}(\nu)e^{i\sum_{j=1}^{\infty}\nu_j\theta_j}. \tag{2.7}$$

Hence, elements f *in* $L^r(\mathbb{T}^\infty, \sigma)$ *are uniquely defined by the Fourier expansion in several complex variables.*

2.2.3 Analytic and harmonic functions in polydiscs

Definition 2.4 (Analytic and harmonic functions in $\mathbb{D}$)**.** *[22, Chapter 3] Recall that a complex valued function f is analytic in $\mathbb{D}$ provided that it is the sum of a convergent power series*

$$f(z) = \sum_{j=0}^{\infty} a_j z^j, \qquad z \in \mathbb{D},$$

which just means that f has a derivative at each point of $\mathbb{D}$.
A complex valued function f on $\mathbb{D}$ is harmonic if it satisfies Laplace's equation:

$$\Delta f(x, y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Any analytic function is a complex-valued harmonic function.

A complex-valued function f is *analytic* in $\mathbb{D}^n$ if it is continuous in $\mathbb{D}^n$ and analytic in each variable separately [34, pp.1]. Denote by $A(\mathbb{D}^n)$ the polydisc algebra which is defined as the class of all continuous complex valued functions on the closure $\overline{\mathbb{D}}^n$ of $\mathbb{D}^n$ whose restriction to $\mathbb{D}^n$ is analytic there. $A(\mathbb{D}^n)$ is closed under pointwise addition and multiplication and is complete with respect to the supremum norm $\|f\|_{\mathbb{D}^n} = \sup_{z \in \mathbb{D}^n} |f(z)|$, $f \in A(\mathbb{D}^n)$.

A continuous complex function f in $\mathbb{D}^n$ is *n-harmonic* if it is harmonic in each complex variable separately: if $z_j = x_j + iy_j \in \mathbb{D}$, f should satisfy the n equations $\Delta_j f = 0$ $(1 \leq j \leq n)$, where $\Delta_j = \partial^2/\partial x_j^2 + \partial^2/\partial y_j^2$. Since the harmonic functions are those for which $\sum_j \Delta_j f = 0$, every n-harmonic function is harmonic in $\mathbb{D}^n$. The two classes coincide if and only if $n = 1$.

2.2.4 Poisson measures on polydiscs

In this section we briefly highlight the development of the Poisson measure $\sigma_\xi^{(n)}$ on $\mathbb{T}^n$ for points $\xi \in \mathbb{D}^n$ (see [10]).

For $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{D}^n$ and $z = (z_1, \ldots, z_n) \in \mathbb{T}^n$ such that $\xi_j = r_j\ e^{i\theta_j}$ and

$z_j = e^{i\varphi_j}$ where $r_j \in [0, 1)$ and $\theta_j, \varphi_j \in [0, 2\pi]$. Let

$$\mathcal{K}(z, \xi) = \prod_{j=1}^{n} \frac{1}{1 - \bar{\xi}_j z_j}$$

be the Szegö kernel for the finite dimensional polydisc. Moreover, the Szegö kernel can be expressed in the series

$$\mathcal{K}(z, \xi) = \sum_{\nu \in (\mathbb{N} \cup \{0\})^n} \bar{\xi}_1^{\nu_1} \dots \bar{\xi}_n^{\nu_n} z_1^{\nu_1} \dots z_n^{\nu_n}, \qquad \nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n.$$

This series is dominated by $\sum_{\nu \in \mathbb{N}^n} |\xi_1|^{\nu_1} \dots |\xi_n|^{\nu_n}$, which converges uniformly on every compact subset of $\mathbb{D}^n$.

The Poisson kernel is expressed in terms of the Szegö kernel by

$$\mathcal{P}_\xi(z) = \frac{|\mathcal{K}(z, \xi)|^2}{\mathcal{K}(\xi, \xi)},$$

which is the product

$$\mathcal{P}_\xi(z) = \mathcal{P}_{r_1}(\theta_1 - \varphi_1) \dots \mathcal{P}_{r_n}(\theta_n - \varphi_n),$$

where $\mathcal{P}_r(\varphi) = (1 - r^2)/(1 - 2r \cos \varphi + r^2)$ is the poisson kernel for the unit disc. Note that for all $z \in \mathbb{T}^n$ and $\xi \in \mathbb{D}^n$ we have $\mathcal{P}_\xi(z) > 0$,

$$\int_{\mathbb{T}^n} \mathcal{P}_\xi(z) \mathrm{d}\sigma_n(z) = 1 \tag{2.8}$$

and

$$\mathcal{P}_\xi(z) = \sum_{\nu \in \mathbb{Z}^n} r_1^{|\nu_1|} \dots r_n^{|\nu_n|} e^{i\nu_1(\theta_1 - \varphi_1)} \dots e^{i\nu_n(\theta_n - \varphi_n)}.$$

For $z = (z_1, \dots, z_n) \in \mathbb{T}^n$, define

$$\begin{aligned} \mathrm{d}\sigma_\xi^{(n)}(z) &= \mathcal{P}_{\xi_1}(z_1) \dots \mathcal{P}_{\xi_n}(z_n) \; \mathrm{d}\sigma_n(z) \\ &= \frac{|\mathcal{K}(z, \xi)|^2}{\mathcal{K}(\xi, \xi)} \mathrm{d}\sigma_n(z), \end{aligned} \tag{2.9}$$

which is the Poisson measure on $\mathbb{T}^n$ for $\xi \in \mathbb{D}^n$. Note that $\sigma_\xi^{(n)}$ is a probability measure and it coincides with the Haar measure σ_n on $\mathbb{T}^n$ whenever $\xi = 0$.

Theorem 2.6. *[34, Theorem 2.1.2] If f is continuous on the closed polydisc $\overline{\mathbb{D}}^n$ and n-harmonic in $\mathbb{D}^n$, then*

$$f(\xi) = \mathcal{P}[f](\xi) := \int_{\mathbb{T}^n} f(z)\mathcal{P}_\xi(z)\mathrm{d}\sigma_n(z), \qquad \forall \xi \in \mathbb{D}^n.$$

It follows that every f with these properties is determined by its values on the distinguished boundary $\mathbb{T}^n$.

If f is any function on $\mathbb{D}^n$ and $0 \le \rho < 1$, f_ρ will denote the function defined on $\mathbb{T}^n$ by: $f_\rho(z) := f(\rho z)$ when $z \in \mathbb{T}^n$, where $\rho z = (\rho z_1, \ldots, \rho z_n)$.

Theorem 2.7. *[34, Theorem 2.1.3]*

(a) If $f \in L^\infty(\mathbb{T}^n)$ and $\xi \in \mathbb{D}^n$, then $|\mathcal{P}[f](\xi)| \le \|f\|_\infty$. Equality for one $\xi \in \mathbb{D}^n$ implies that f is constant a.e. on $\mathbb{T}^n$.

(b) If $f \in C(\mathbb{T}^n)$ then $\mathcal{P}[f]$ extends to a continuous function on $\overline{\mathbb{D}}^n$.

(c) If $r \in [1, \infty)$, $f \in L^r(\mathbb{T}^n)$ and $u = \mathcal{P}[f]$, then $\|u_\rho\|_{L^r(\mathbb{T}^n)} \le \|f\|_{L^r(\mathbb{T}^n)}$ and $\|u_\rho - f\|_{L^r(\mathbb{T}^n)} \to 0$ as $\rho \to 1$.

By virtue of Theorem 2.7(a), every $f \in A(\mathbb{D}^n)$ satisfies the maximum principle $\|f\|_{\mathbb{D}^n} = \|f\|_{\mathbb{T}^n}$.

The following characterises analytic Poisson integrals in terms of Fourier coefficients and features the restriction of members of $A(\mathbb{D}^n)$ to $\mathbb{T}^n$.

The Fourier coefficients $\widehat{f}(\nu)$ of a function on $\mathbb{T}^n$ are defined as:

$$\widehat{f}(\nu) = \int_{\mathbb{T}^n} f(z)\ \bar{z}^\nu \mathrm{d}\sigma_n(z), \qquad \nu \in \mathbb{Z}^n,$$

where $\bar{z}^\nu$ is an abbreviation for $\bar{z}_1^{\nu_1} \ldots \bar{z}_n^{\nu_n}$.

Theorem 2.8. *[34, Theorem 2.2.1] A function $f \in C(\mathbb{T}^n)$ is the restriction of a member of $A(\mathbb{D}^n)$ if, and only if, $\widehat{f}(\nu) = 0$ outside $\mathbb{N}^n$.*

Definition 2.5 (Hardy spaces on $\mathbb{T}^n$)**.** *For $r \in [1, \infty)$. The Hardy space $H^r(\mathbb{T}^n)$ is defined as the analytic subspace of the space $L^r(\mathbb{T}^n, \sigma_n)$, i.e.*

$$H^r(\mathbb{T}^n) = \left\{ f \in L^r(\mathbb{T}^n, \sigma_n) : \ f(z) = \sum_{\mu \in \mathbb{N}^n} a_\mu z_1^{\mu_1} \cdots z_n^{\mu_n} \right\}.$$

This space is Banach when endowed with the norm

$$\|f\|_{H^r(\mathbb{T}^n)} := \left(\int_{\mathbb{T}^n} |f(z)|^r \mathrm{d}\sigma_n(z) \right)^{1/r}.$$

For $r = 2$, it is a Hilbert space in the norm

$$\|f\|_{H^2(\mathbb{T}^n)} := \left(\sum_{\mu \in \mathbb{N}^n} |a_\mu|^2 \right)^{1/2}.$$

In the particular case $r = \infty$, we denote by $H^\infty(\mathbb{T}^n)$ the space of all bounded analytic functions on $\mathbb{T}^n$ which is a Banach space when equipped with the norm

$$\|f\|_\infty = \sup_{z \in \mathbb{T}^n} |f(z)| \qquad \textit{for} \qquad f \in H^\infty(\mathbb{T}^n).$$

Definition 2.6 (Hardy spaces on $\mathbb{D}^n$)**.** *For $r \in [1, \infty)$. The Hardy space $H^r(\mathbb{D}^n)$ is defined as the class of all analytic functions f in $\mathbb{D}^n$ such that $\|f\|_{H^r(\mathbb{D}^n)} < \infty$, where*

$$H^r(\mathbb{D}^n) = \left\{ f : f \in A(\mathbb{D}^n) \ \textit{such that} \ \|f\|_{H^r(\mathbb{D}^n)} < \infty \right\}$$

and

$$\|f\|_{H^r(\mathbb{D}^n)} := \left(\sup_{0 \le \rho < 1} \int_{\mathbb{T}^n} |f_\rho(z)|^r \mathrm{d}\sigma_n(z) \right)^{1/r},$$

(since $|f_\rho|^r$ is n-harmonic, $\sup$ *can be replaced by limit as $\rho \to 1$).*

$H^\infty(\mathbb{D}^n)$ is the space of all bounded analytic functions in $\mathbb{D}^n$ such that

$$\|f\|_\infty = \sup_{z \in \mathbb{D}^n} |f(z)| \qquad \textit{for} \qquad f \in H^\infty(\mathbb{D}^n).$$

In particular, $H^2(\mathbb{D}^n)$ is defined as follows: if $f(z) = \sum_{\mu \in \mathbb{N}^n} a_\mu z_1^{\mu_1} \ldots z_n^{\mu_n}$, then

$f \in H^2(\mathbb{D}^n)$ if and only if $\sum_{\mu\in\mathbb{N}^n} |a_\mu|^2 < \infty$. In fact,

$$\|f\|_{H^2(\mathbb{D}^n)} := \left(\sum_{\mu\in\mathbb{N}^n} |a_\mu|^2\right)^{1/2}.$$

An extremely useful feature of the one variable theory is that any function $f \in H^r(\mathbb{T})$ can be extended to an analytic function on the open unit disc $\mathbb{D}$. This remains true in finite dimensions with almost no restriction to the radial or even the non-tangential approach.

Theorem 2.9 (Fatou's Theorem)**.** *[40, Theorem XV11-4.8] If $f \in H^r(\mathbb{D}^n)$, then f has a non-tangential limit at almost all points of the distinguished boundary $\mathbb{T}^n$.*

Remark 2.1. *This property of the Hardy spaces of the finite dimensional torus $\mathbb{T}^n$ ensures the isometric embedding between the two spaces $H^r(\mathbb{D}^n)$ and $H^r(\mathbb{T}^n)$. Thus the restriction $f \mapsto f|_{\mathbb{T}^n}$ as a map from $H^r(\mathbb{D}^n)$ to $H^r(\mathbb{T}^n) \subset L^r(\mathbb{T}^n, \sigma_n)$ makes sense as a consequence of Fatou's Theorem. Moreover, for every $f \in H^r(\mathbb{D}^n)$ the radial limit $\lim\limits_{\rho\to 1^-} f(\rho z)$ exists at almost all $z \in \mathbb{T}^n$.*

The following inequality from [38] will be used in Chapter 7 when calculating the norm of the multiplication operator on Hardy spaces $H^r(\mathbb{D}^n)$, $r \in (1,\infty)$. For all $\xi \in \mathbb{D}^n$ we have

$$|f(\xi)|^r \leq \prod_{j=1}^{n} \frac{1}{1-|\xi_j|^2} \, \|f\|^r_{H^r(\mathbb{D}^n)}, \qquad \forall f \in H^r(\mathbb{D}^n). \tag{2.10}$$

(see also Theorem 6.1 in [10] for the infinite dimensional case). This inequality is sharp for the function

$$h_\xi(z) = \prod_{j=1}^{n} \frac{(1-|\xi_j|^2)^{1/r}}{|1-\bar{\xi}_j z_j|^{2/r}} = (\mathcal{P}_\xi(z))^{1/r}, \quad \xi, z \in \mathbb{D}^n. \tag{2.11}$$

Observe that $\|h_\xi\|_{H^r(\mathbb{D}^n)} = 1$ (see statement (2.8)).

In order to state the definition of the space $H^r(\mathbb{T}^\infty)$ for $r \in [1,\infty]$, observe that $\mathbb{T}^\infty$ is a compact abelian group with dual $\mathbb{Z}_0^\infty$ and a normalised Haar measure σ. Elements f in $L^r(\mathbb{T}^\infty, \sigma)$ are uniquely defined by their Fourier series expansion (given by (2.7))

$$f(z) = \sum_{\mu \in \mathbb{Z}_0^\infty} a_\mu z_1^{\mu_1} \dots z_k^{\mu_k},$$

where the Fourier coefficients are defined in the standard manner (2.6) and $\mu \in \mathbb{Z}_0^\infty$ means that only finitely many of the components of the index sequence μ are non-zero.

Definition 2.7 (Hardy spaces on $\mathbb{T}^\infty$)**.** *For $r \in [1, \infty)$. Define the Hardy spaces $H^r(\mathbb{T}^\infty)$ to be the analytic part of L^r in the following way*

$$H^r(\mathbb{T}^\infty) = \left\{ f \in L^r(\mathbb{T}^\infty, \sigma) : f(z) = \sum_{\mu \in \mathbb{N}_0^\infty} a_\mu z_1^{\mu_1} \dots z_k^{\mu_k} \right\}.$$

This is a Banach space in the norm

$$\|f\|_{H^r(\mathbb{T}^\infty)} := \left(\int_{\mathbb{T}^\infty} |f(z)|^r \mathrm{d}\sigma(z) \right)^{1/r}$$

and a Hilbert space in the case $r = 2$ endowed with the norm

$$\|f\|_{H^2(\mathbb{T}^\infty)} := \left(\int_{\mathbb{T}^\infty} |f(z)|^2 \mathrm{d}\sigma(z) \right)^{1/2} = \left(\sum_{\mu \in \mathbb{N}_0^\infty} |a_\mu|^2 \right)^{1/2},$$

the latter is due to Parseval's identity.

The case $r = \infty$ defines the Banach space $H^\infty(\mathbb{T}^\infty)$ of all bounded analytic functions on the infinite dimensional torus in the norm

$$\|f\|_\infty = \sup_{z \in \mathbb{T}^\infty} |f(z)|.$$

Note 2.1. *It is no longer straight forward to extend functions $f \in H^r(\mathbb{T}^\infty)$ to functions on the polydisc $\mathbb{D}^\infty$. This is because point evaluations for functions in the Hardy space of the polydisc are well defined only in $l^2 \cap \mathbb{D}^\infty$ for $r < \infty$ and in $c_0 \cap \mathbb{D}^\infty$ for $r = \infty$, see [10], where c_0 is the closed subspace of l^∞ of sequences that converge to zero.*

2.2.5 Hardy-Dirichlet spaces [20]

This section aims at showing the isometric isomorphic correspondence between the Hardy spaces of analytic functions on the polycircle $H^r(\mathbb{T}^\infty)$ and the Hardy spaces of Dirichlet series $\mathscr{H}^r$ for all $r \in (1, \infty)$. This aspect is important in the study of bases properties of the system of dilations of monomials, $\mathfrak{H}_\varphi$, defined in Chapter 7.

The space

$$\mathscr{H} = \left\{ \sum_{n=1}^{\infty} a_n n^{-s}, \ \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\},$$

was first introduced in [20]. It is a Hilbert space of Dirichlet series with square summable coefficients. Then the non-Hilbertian case $\mathscr{H}^r$, $r \in [1, \infty)$ was described in [4] (see (2.12) below). For $r = 2$, $\mathscr{H}^2 = \mathscr{H}$.

To understand the configuration of these spaces, we need a group-theoretical identification of $\mathbb{T}^\infty$, which we shall now describe. Let $\mathscr{E}$ be the dual group of $\mathbb{Q}_+$, where $\mathbb{Q}_+$ denotes the multiplicative discrete group of strictly positive rational numbers. $\mathscr{E}$ is the set of characters $\mathcal{X} : \mathbb{Q}_+ \longrightarrow \mathbb{C}$, such that

(a) $\mathcal{X}(mn) = \mathcal{X}(m)\mathcal{X}(n)$ for all $m, n \in \mathbb{Q}_+$.

(b) $|\mathcal{X}(n)| = 1$ for all $n \in \mathbb{Q}_+$.

Given a point $z = (z_1, z_2, \ldots) \in \mathbb{T}^\infty$, define $\mathcal{X}$ at the primes through

$$\mathcal{X}(2) = z_1, \ \mathcal{X}(3) = z_2, \ \ldots, \ \mathcal{X}(p_k) = z_k, \ \ldots$$

and extend the definition multiplicatively. This identification yields a character. The set of all characters is then obtained by the same procedure. This identification is topological and the Haar measures on $\mathbb{T}^\infty$ and $\mathscr{E}$ are identical (for more details see [20] pages 8-9).

For general $r \in [1, \infty)$, the spaces $\mathscr{H}^r$ are defined to be the closure of Dirichlet polynomials in the norm

$$\left(\lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} \left| \sum_{n=1}^{N} a_n n^{-it} \right|^r \mathrm{d}t \right)^{1/r}. \tag{2.12}$$

This norm can be understood as the ergodic theorem on the infinite-dimensional torus $\mathbb{T}^\infty$. To briefly explain this, we note that $\mathbb{T}^\infty$ is a compact Abelian group with a Haar measure σ and has a dual group $\mathbb{Z}_0^\infty$. By the Fourier analysis on groups, $f \in H^r(\mathbb{T}^\infty)$ has a Fourier expansion $f(z) = \sum_{\nu \in \mathbb{N}_0^\infty} a_\nu z^\nu$ where $z \in \mathbb{T}^\infty$ and $z^\nu = z_1^{\nu_1} \dots z_k^{\nu_k}$ for $\nu = (\nu_j)_{j\in\mathbb{N}} \in \mathbb{N}_0^\infty$. The central observation, essentially called Kronecker's Lemma, is that the path $\phi : t \mapsto (2^{-it}, \dots, p_i^{-it}, \dots)$, where p_i is the i-th prime number, is ergodic in $\mathbb{T}^\infty$. Then by the Ergodic Theorem for continuous functions f on $\mathbb{T}^\infty$, we have

$$\left(\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} |f \circ \phi(t)|^r \mathrm{d}t\right)^{1/r} = \|f\|_{H^r(\mathbb{T}^\infty)}.$$

Note that $f \circ \phi$ is a Dirichlet series provided that we identify a_ν with a_n when $n = p_1^{\nu_1} p_2^{\nu_2} \dots$. By the uniqueness of prime number factorisation, the map from $H^r(\mathbb{T}^\infty)$ to $\mathscr{H}^r$ given by $f \mapsto f \circ \phi$ has an inverse, which is called the Boher lift [1]. The construction of the $\mathscr{H}^r$ spaces is mainly based on Boher's observation which gives strong links between Dirichlet series and Fourier series on the infinite dimensional torus $\mathbb{T}^\infty$. Let $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ be a Dirichlet series. Decompose each integer into a product of prime factors, $n = \mathfrak{p}_1^{\nu_1(n)} \cdots \mathfrak{p}_k^{\nu_k(n)}$, and set $z = (\mathfrak{p}_1^{-s}, \mathfrak{p}_2^{-s}, \cdots)$. Then,

$$f(s) = \sum_{n=1}^{\infty} a_n \left(\mathfrak{p}_1^{-s}\right)^{\nu_1(n)} \cdots \left(\mathfrak{p}_k^{-s}\right)^{\nu_k(n)} = \sum_{\nu \in \mathbb{N}_0^\infty} a_\nu z_1^{\nu_1(n)} \cdots z_k^{\nu_k(n)} =: \mathscr{D}f(z),$$

where a_n corresponds to a_ν for all $n \in \mathbb{N}$ and $\nu \in \mathbb{N}_0^\infty$. The Dirichlet series becomes a Fourier series in infinitely many variables indexed by $\mathbb{N}_0^\infty$. This identification preserves both topological and measure properties.

Denote by $\mathscr{P}$ the set of all Dirichlet polynomials, $P(s) = \sum_{n=1}^N a_n n^{-s}$. If $P \in \mathscr{P}$, define

$$\|P\|_{\mathscr{P}}^r = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} |P(it)|^r \mathrm{d}t.$$

Denote its corresponding trigonometric polynomial by $\mathscr{D}P(z) = \sum_{\nu\in\mathbb{N}^k} a_\nu z_1^{\nu_1} \cdots z_k^{\nu_k}$, then

$$\sup_{Re(s)\geq 0} |P(s)| = \sup_{z\in\overline{\mathbb{D}}^k} |\mathscr{D}P(z)|.$$

Lemma 2.11. *[4, Lemma 3] For $P \in \mathscr{P}$, then*

$$\|P\|_{\mathscr{P}} = \|\mathscr{D}P\|_{H^r(\mathbb{T}^\infty)}.$$

Definition 2.8. *[4] The space $\mathscr{H}^r$ is the completion of the set $\mathscr{P}$ with respect to the norm $\|\cdot\|_{\mathscr{P}}$. In fact $\mathscr{H}^r$ can be identified by the space $H^r(\mathbb{T}^\infty)$ for all $r \in [1, \infty)$.*

Lemma 2.12. *[4, Theorem 2] The operator $\mathscr{D} : \mathscr{P} \longrightarrow H^r(\mathbb{T}^\infty)$ extends to an isometric isomorphism from $\mathscr{H}^r$ onto $H^r(\mathbb{T}^\infty)$.*

A fundamental corollary regarding bases properties of the set $\{n^{-s}\}_{n\in\mathbb{N}}$ was settled in [1] via the following theorem.

Theorem 2.10. *[1, Corollary 4] Let $r \in (1, \infty)$. The set of functions n^{-s}, $n \in \mathbb{N}$ forms an orthonormal basis of $\mathscr{H}$ and a Schauder basis of $\mathscr{H}^r$.*

Boher's observation [1] implies the following in the context of Hardy spaces on $\mathbb{T}^\infty$.

Theorem 2.11. *The set of monomials $\{\mathfrak{h}_n\}_{n\in\mathbb{N}}$ where*

$$\mathfrak{h}_n(z) = z_1^{\nu_1(n)} \dots z_k^{\nu_k(n)}, \quad (z \in \mathbb{T}^\infty,\ \nu = (\nu_j)_{j\in\mathbb{N}} \in \mathbb{N}_0^\infty), \tag{2.13}$$

forms an orthonormal basis of the Hardy space $H^2(\mathbb{T}^\infty)$ and a Schauder basis of $H^r(\mathbb{T}^\infty)$ whenever $r \in (1, \infty)$.

Proof. This is a direct consequence of the combination of Theorem 2.10 and Lemma 2.12. $\square$

Chapter 3

Criteria for Bases Properties of Dilated Functions in Lebesgue Spaces

It is a familiar fact that the classical sine functions $\mathfrak{E}_{\sin}$ and the classical cosine functions $\mathfrak{E}_{\cos}$ form bases of the Lebesgue spaces $L^r(0,1)$ for all $r \in (1,\infty)$. For the sine functions this follows from the classical result of M. Riesz (see, Lemma 2.1), while the cosine functions have the claimed property follows via an even extension.

In this chapter we shall present criteria for demonstrating bases properties of the family $\mathfrak{E}_f$, introduced in Chapter 1, which does not have the property of orthogonality. Thus we are forced to give up on the very useful Parseval's relation, and seek other types of methods to handle the problem. One of those is the idea of *stability* of sequences in a Banach space, through which we shall find that the bases properties of sequences can sometimes be deduced from the fact that they are "near" in a sense or "equivalent" to a sequence already known to posses the required property.

This suggests deriving various properties of the change of coordinates maps that take the basis $\mathfrak{E}_e := \{e_n(\cdot)\}_n \subset L^r(0,1)$ into the sequence $\mathfrak{E}_f$, as shown in Section 3.1 below.

3.1 The change of coordinates map

Given any $g \in L^r(0,1)$, denote the odd extension[1] of g with respect to 1 by

$$\tilde{g}(x) = \begin{cases} g(x) & x \in [0,1] \\ -g(2-x) & x \in (1,2]. \end{cases}$$

A 2−periodic extension of g to the whole of $\mathbb{R}$ is then written as

$$g^*(x) = \tilde{g}\left(x - 2\left\lfloor \frac{x}{2} \right\rfloor\right). \tag{3.1}$$

The floor function $\lfloor y \rfloor \in \mathbb{Z}$ is the unique integer such that $y - \lfloor y \rfloor \in [0,1)$. For any $n \in \mathbb{N}$, let

$$M_n g(x) := g^*(nx).$$

Lemma 3.1. *The operators $M_n : L^r(0,1) \longrightarrow L^r(0,1)$ are linear isometries.*

Proof. Indeed,

$$\begin{aligned}
\|M_n g\|^r_{L^r(0,1)} &= \int_0^1 |M_n g(x)|^r \mathrm{d}x = \int_0^1 |g^*(nx)|^r \mathrm{d}x = \int_0^1 \left|\tilde{g}\left(nx - 2\left\lfloor \frac{nx}{2} \right\rfloor\right)\right|^r \mathrm{d}x \\
&= \frac{1}{n}\int_0^n \left|\tilde{g}\left(y - 2\left\lfloor \frac{y}{2} \right\rfloor\right)\right|^r \mathrm{d}y = \frac{1}{n}\sum_{l=0}^{n-1}\int_l^{l+1} \left|\tilde{g}\left(y - 2\left\lfloor \frac{y}{2} \right\rfloor\right)\right|^r \mathrm{d}y \\
&= \frac{1}{n}\left[\sum_{\substack{l=0 \\ l\equiv_2 0}}^{n-1}\int_l^{l+1} \left|\tilde{g}\left(y - 2\left\lfloor \frac{y}{2} \right\rfloor\right)\right|^r \mathrm{d}y + \sum_{\substack{l=1 \\ l\equiv_2 1}}^{n-1}\int_l^{l+1} \left|\tilde{g}\left(y - 2\left\lfloor \frac{y}{2} \right\rfloor\right)\right|^r \mathrm{d}y\right].
\end{aligned}$$

Changing variables to $w = y - l$ for $l \equiv_2 0$ and $z = y - (l-1)$ for $l \equiv_2 1$, gives

$$\left\lfloor \frac{y}{2} \right\rfloor = \begin{cases} \frac{l}{2} & \text{whenever } l \equiv_2 0 \\ \frac{l-1}{2} & \text{whenever } l \equiv_2 1. \end{cases}$$

Here and elsewhere below we will write $j \equiv_2 k$ to denote that $j \equiv k \pmod 2$.

[1] See Remark 3.2 below.

Hence,

$$\|M_n g\|_{L^r(0,1)}^r = \frac{1}{n}\left[\sum_{\substack{l=0 \\ l\equiv_2 0}}^{n-1}\int_0^1 |g(w)|^r \mathrm{d}w + \sum_{\substack{l=1 \\ l\equiv_2 1}}^{n-1}\int_1^2 |\tilde{g}(z)|^r \mathrm{d}z\right].$$

Another change of variables $z = 2 - w$, then yields

$$\|M_n g\|_{L^r(0,1)}^r = \frac{1}{n}\left[n\int_0^1 |g(w)|^r \mathrm{d}w\right] = \|g\|_{L^r(0,1)}^r$$

as claimed. □

Moreover

$$M_1 = \mathrm{Id} \qquad \text{and} \qquad M_{jk} = M_j M_k \qquad \forall j,k \in \mathbb{N}. \tag{3.2}$$

Let $e_j(x) = \sin(j\pi x)$. Let $f \in L^r(0,1)$ with an odd extension to $L^r(-1,1)$ and Fourier coefficients

$$\widehat{f}_j = 2\int_0^1 f(x)\sin(j\pi x)\mathrm{d}x \quad \text{such that} \quad \sum_{j=1}^{\infty} |\widehat{f}_j| < \infty. \tag{3.3}$$

Let $f_n \in L^r(0,1)$ defined as $f_n(x) := f^*(nx)$, for all $n \in \mathbb{N}$. Since the functions f_n are continuous on $[0,1]$ then,

$$f_n(x) = \sum_{k=1}^{\infty} \widehat{f}_n(k) e_k(x) \quad \text{such that} \quad \widehat{f}_n(k) = 2\int_0^1 f_n(x) e_k(x)\mathrm{d}x.$$

The infinite series is both pointwise convergent for all $x \in [0,1]$ (Theorem 2.1) and also convergent in the norm of $L^r(0,1)$ for all $r \in (1,\infty)$ (Lemma 2.1). Since f_1 is symmetric about $x = \frac{1}{2}$, for every even k we have $\widehat{f_1}(k) = 0$. For all $n > 1$,

$$\begin{aligned}
\widehat{f_n}(k) &= 2\int_0^1 f_1(nx)\sin(k\pi x)\mathrm{d}x \\
&= 2\int_0^1 \left(\sum_{m=1}^{\infty} \widehat{f}_m \sin(m\pi nx)\right)\sin(k\pi x)\mathrm{d}x \\
&= 2\sum_{m=1}^{\infty} \widehat{f}_m \int_0^1 \sin(mn\pi x)\sin(k\pi x)\mathrm{d}x \\
&= \begin{cases} \widehat{f}_m & \text{for } mn = k,\ m \equiv_2 1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Here we can exchange the infinite summation with the integral sign, due to the pointwise convergence of the series, statement (3.3) and the Dominated Convergence theorem [3, Proposition 2.17, pp. 37].

Let

$$T = \sum_{j=1}^{\infty} \widehat{f}_j M_j. \tag{3.4}$$

Lemma 3.2. *The operator $T : L^r(0,1) \longrightarrow L^r(0,1)$ is bounded and linear. Moreover,*

$$Te_n = f_n \qquad \forall n \in \mathbb{N}.$$

Proof. By virtue of (3.3), Lemma 3.1 and the triangle inequality, it follows that the expression (3.4) is convergent in the operator norm of $L^r(0,1)$ and that

$$\|T\|_{\mathcal{B}(L^r(0,1))} \leq \sum_{j=1}^{\infty} |\widehat{f}_j| \|M_j\|_{\mathcal{B}(L^r(0,1))} = \sum_{j=1}^{\infty} |\widehat{f}_j| < \infty.$$

In addition,

$$Te_n = \sum_{j=1}^{\infty} \widehat{f}_j M_j e_n = \sum_{j=1}^{\infty} \widehat{f}_j e_{nj} = \sum_{k=1}^{\infty} \widehat{f_n}(k) e_k = f_n \qquad \forall n \in \mathbb{N}.$$

Linearity is trivial. ∎

3.2 Criteria for bases properties of dilated functions in $L^r(0,1)$

In this section we shall present criteria for determining bases properties of the sequence $\mathfrak{E}_f$. These criteria involve the idea of "equivalence" between two sequences in a Banach space.

According to Lemmas 2.1, 3.2 and Theorem 2.2, $\mathfrak{E}_f$ is a Schauder basis of $L^r(0,1)$ if and only if the operator $T : L^r(0,1) \longrightarrow L^r(0,1)$ in (3.4) is a homeomorphism, cf. [21] or [37]. This approach enables f_n to inherit from e_n the property of being a basis in $L^r(0,1)$ for all $r \in (1,\infty)$ and $n \in \mathbb{N}$. In turns, it provides together with [23, Theorem IV-1.16] a criterion, cf. [5], [11], [12], [7] and [8], that measures the bases properties of this set. We describe this in two different cases in sections 3.2.1 and 3.2.2.

Let $\mathcal{F} \subset \mathbb{N}$ be finite such that $1 \in \mathcal{F}$. Write the operator T, given by (3.4), as $T = M + V$ where

$$M = \sum_{j\in\mathcal{F}} \widehat{f}_j M_j \qquad \text{and} \qquad V = \sum_{j\in\mathbb{N}\setminus\mathcal{F}} \widehat{f}_j M_j. \tag{3.5}$$

Then,

$$\left.\begin{aligned} &M^{-1} : \text{exists} \\ &\|V\| < \|M^{-1}\|^{-1} \end{aligned}\right\} \quad \Rightarrow \quad \begin{cases} T = M + V = M[\mathrm{Id} + M^{-1}V] \\ \text{is a homeomorphism.} \end{cases} \tag{3.6}$$

As we shall see in the forthcoming chapters, it can be very difficult to determine whether T is invertible or not even for simple functions f. Criteria, such as those formulated in [5, Section 4] and [7, Sections 4 and 7], give sufficient conditions for invertibility in Banach and Hilbert spaces respectively, provided we have sharp estimates on $|\widehat{f_k}|$ both for large k and small k. Corollary 3.1 for the case $\mathcal{F} = \{1\}$ and Corollary 3.2 for the case $\mathcal{F} \neq \{1\}$ generalise these criteria. The emphasis here is the computability, either analytical or by accurate numerical means, of all the quantities involved.

In this section we distinguish between the two cases: $\mathcal{F} = \{1\}$, when the operator M includes only *one-term*; $\mathcal{F} \neq \{1\}$, when M is expressed as a finite *multi-term* linear combination of the isometries M_j.

3.2.1 One-term Schauder basis criteria ($\mathcal{F} = \{1\}$)

In this case, $M = \widehat{f}_1 \,\mathrm{Id}$. If $\widehat{f}_1 \neq 0$, then M is invertible and

$$M^{-1} = \widehat{f}_1^{-1} \,\mathrm{Id}\,. \tag{3.7}$$

Theorem 3.1. *If $\widehat{f}_1 \neq 0$ and*

$$\sum_{j=2}^{\infty} |\widehat{f}_j| < |\widehat{f}_1|, \tag{3.8}$$

then $\mathfrak{E}_f$ is a Schauder basis of $L^r(0,1)$ for all $r \in (1, \infty)$.

Proof. The operator $T = M + V$, given in (3.4), is a bounded linear operator mapping e_n into f_n (see, Lemma 3.2). In order to show that the set $\mathfrak{E}_f$ is a Schauder basis in $L^r(0,1)$ it is sufficient to show that the operator T is bijective. Notice that,

$$\|V\| \leq \sum_{j=2}^{\infty} |\widehat{f}_j|.$$

According to (3.7) and (3.8), $\|V\| < \|M^{-1}\|^{-1}$. Then in view of (3.6), the result follows. $\square$

Assume that the Fourier coefficients of f are such that

$$|\widehat{f}_j| \leq \phi_j, \qquad \forall j \in \mathbb{N} \tag{3.9}$$

for a sequence $\{\phi_j\}_{j=1}^{\infty} \in \ell^1(\mathbb{N})$. We set $\Phi = \sum_{j=1}^{\infty} \phi_j$.

Corollary 3.1. *Let $k \in \mathbb{N}$ be fixed. If*

$$2|\widehat{f}_1| - \Phi + \sum_{j=1}^{k} \Big(\phi_j - |\widehat{f}_j|\Big) > 0, \tag{3.10}$$

then $\mathfrak{E}_f$ *is a Schauder basis of* $L^r(0,1)$ *for all* $r \in (1,\infty)$.

Proof.

$$\begin{aligned} \|V\| &\leq \sum_{j=2}^{\infty} |\widehat{f}_j| = \sum_{j=1}^{\infty} |\widehat{f}_j| - |\widehat{f}_1| \\ &\leq \sum_{j=1}^{k} |\widehat{f}_j| + \sum_{j=k+1}^{\infty} \phi_j - |\widehat{f}_1| \\ &= \sum_{j=1}^{k} \left(|\widehat{f}_j| - \phi_j \right) + \Phi - |\widehat{f}_1|. \end{aligned}$$

According to (3.10), $\|V\| < |\widehat{f}_1|$. Then, in view of statements (3.7) and (3.6), the operator T is a homeomorphism and the family $\mathfrak{E}_f$ is a Schauder basis of $L^r(0,1)$, $r \in (1,\infty)$. □

3.2.2 Multi-term Riesz basis criteria ($\mathcal{F} \neq \{1\}$)

Let $\mathcal{F} \subset \mathbb{N}$ be finite such that $1 \in \mathcal{F}$. Let $\mathbb{P}(\mathbb{N}) \subset \mathbb{N}$ denote the set of all prime numbers not including 1. Set

$$\mathbb{P}(\mathcal{F}) := \{\mathfrak{p} \in \mathbb{P}(\mathbb{N}) : \mathfrak{p}|n \text{ for some } n \in \mathcal{F}\}. \tag{3.11}$$

For $n \in \mathcal{F}$, we consider prime factorisations of the form

$$n = \prod_{\mathfrak{p} \in \mathbb{P}(\mathcal{F})} \mathfrak{p}^{\nu_{\mathfrak{p}}(n)}$$

where $\mathbb{P}(\mathcal{F})$ is as in (3.11) and the exponent $\nu_{\mathfrak{p}}(n) = 0$ whenever $\mathfrak{p}$ does not divide n. Let $d = \#\mathbb{P}(\mathcal{F})$, that is, the number of primes dividing $n \in \mathcal{F}$, and order the elements of $\mathbb{P}(\mathcal{F})$ in an increasing manner

$$\mathbb{P}(\mathcal{F}) = \{\mathfrak{p}_j\}_{j=1}^{d}.$$

Then

$$n = \mathfrak{p}_1^{\nu_{\mathfrak{p}_1}(n)} \cdots \mathfrak{p}_d^{\nu_{\mathfrak{p}_d}(n)}, \qquad \forall n \in \mathcal{F},$$

and according to (3.2) the operator M in (3.5) takes the form

$$\begin{aligned} M &= \sum_{n \in \mathcal{F}} \widehat{f}_n M_n \\ &= \sum_{n \in \mathcal{F}} \widehat{f}_n M_{\mathfrak{p}_1}^{\nu_{\mathfrak{p}_1}(n)} \cdots M_{\mathfrak{p}_d}^{\nu_{\mathfrak{p}_d}(n)} \\ &= \sum_{\nu \in \mathbb{N}^d} \widehat{f}_n M_{\mathfrak{p}_1}^{\nu_{\mathfrak{p}_1}(n)} \cdots M_{\mathfrak{p}_d}^{\nu_{\mathfrak{p}_d}(n)}. \end{aligned} \tag{3.12}$$

These new criteria are built upon those of [7, Corollary 4.3 and Corollary 4.4] where $\mathcal{F} = \{1, \mathfrak{p} = 3, \mathfrak{p}^2 = 9\}$, that is, the *multi-term* criteria is generalising the outcomes of the literature [7] to the case of any prime $\mathfrak{p} \in \mathbb{P}(\mathcal{F})$ where $\mathcal{F} = \{1, \mathfrak{p}, \mathfrak{p}^2\}$. More than that they study the case when a finite number of primes $\mathfrak{p}_j \in \mathbb{P}(\mathcal{F})$ $(j = 1, \ldots, d)$ are considered such that $\mathcal{F} = \{1, \mathfrak{p}_1, \mathfrak{p}_1^2, \ldots, \mathfrak{p}_d, \mathfrak{p}_d^2\}$.

In the notation of Section 2.2. Set $z = (z_{\mathfrak{p}_1}, \ldots, z_{\mathfrak{p}_d}) \in \mathbb{T}^d$. Replacing $M_{\mathfrak{p}_j}$ by $z_{\mathfrak{p}_j}$ in (3.12), we obtain a *symbol*

$$m(z) = \sum_{n \in \mathcal{F}} \widehat{f}_n z_{\mathfrak{p}_1}^{\nu_{\mathfrak{p}_1}(n)} \cdots z_{\mathfrak{p}_d}^{\nu_{\mathfrak{p}_d}(n)} \in H^\infty(\mathbb{T}^d), \tag{3.13}$$

(see Section 7.1 for further information). Moreover,

$$\|m\|_\infty = \sup_{z \in \mathbb{T}^d} |m(z)|. \tag{3.14}$$

Note 3.1. *The symbol m is a polynomial in several (finite) complex variables, i.e., $m \in H^\infty(\mathbb{T}^d)$ with $d < \infty$. However in some context of this thesis we will allow, by an abuse of notation, m to be considered as a power series in infinitely many variables (i.e., $m \in H^\infty(\mathbb{T}^\infty)$) but with multiplicities $\nu_{\mathfrak{p}_j} = 0$ for positive integers $j > d$. This means that the new components $z_{\mathfrak{p}_j}$ $(j > d)$ of $z \in \mathbb{T}^\infty$ will not contribute to the calculation of $\|m\|_\infty$ that is,*

$$\|m\|_\infty = \sup_{z \in \mathbb{T}^d} |m(z)| = \sup_{z \in \mathbb{T}^\infty} |m(z)|.$$

Define the operator $\mathscr{M}_m : H^2(\mathbb{T}^\infty) \longrightarrow H^2(\mathbb{T}^\infty)$ such that

$$\mathscr{M}_m F(z) = m(z)F(z), \qquad \forall F \in H^2(\mathbb{T}^\infty).$$

Observe that

$$\|\mathscr{M}_m\|_{\mathcal{B}(H^2(\mathbb{T}^\infty))} = \|m\|_\infty, \tag{3.15}$$

which is an immediate consequence of Theorem 7.3 when $\varphi = m$, $\mathcal{T}_\varphi = \mathscr{M}_m$ and $r = 2$.

Remark 3.1. *Below we describe the connection between the operator M and the multiplication operator $\mathscr{M}_m$ by means of the isometric isomorphic correspondence between the Lebesgue spaces $L^2(0,1)$ and the Hardy spaces $H^2(\mathbb{T}^\infty)$. The study is formulated for functions from the space $L^2(0,1)$ with odd extensions to the whole real line. A similar argument is also valid when considering functions in $L^2(0,1)$ with cosine Fourier expansions (see Remark 3.2 for more details).*

Given a function $g \in L^2(0,1)$ with the Fourier sine expansion

$$g(x) = \sum_{j=1}^{\infty} \widehat{g}_j e_j(x) \quad \text{such that} \quad \widehat{g}_j = 2\int_0^1 g(x)e_j(x)\mathrm{d}x. \tag{3.16}$$

Notice that its Fourier coefficients satisfy

$$\sum_{j=1}^{\infty} |\widehat{g}_j|^2 < \infty$$

due to Parseval's identity and Lemma 2.1.

The series (3.16) can also be presented in terms of the isometries M_j in the form

$$g(x) = \sum_{j=1}^{\infty} \widehat{g}_j M_j e_1(x).$$

Decompose j into its unique prime factors $j = \mathfrak{q}_1^{\mu_{\mathfrak{q}_1}(j)} \cdots \mathfrak{q}_s^{\mu_{\mathfrak{q}_s}(j)}$. According to (3.2),

$$g(x) = \sum_{\mu \in \mathbb{N}_0^\infty} \widehat{g}_j M_{\mathfrak{q}_1}^{\mu_{\mathfrak{q}_1}(j)} \cdots M_{\mathfrak{q}_s}^{\mu_{\mathfrak{q}_s}(j)} e_1(x).$$

Replace the $M_{\mathfrak{q}_j}$ by $z_{\mathfrak{q}_j}$ so that the series in (3.16) formally takes the form

$$Ug(z) := \sum_{\mu \in \mathbb{N}_0^\infty} \widehat{g}_j z_{\mathfrak{q}_1}^{\mu_{\mathfrak{q}_1}(j)} \cdots z_{\mathfrak{q}_s}^{\mu_{\mathfrak{p}_s}(j)} \in H^2(\mathbb{T}^\infty).$$

From now on, for a given element $g \in L^2(0,1)$, Ug denotes the corresponding power series in the Hardy space $H^2(\mathbb{T}^\infty)$. Observe that due to Definition 2.7 of $H^2(\mathbb{T}^\infty)$ we conclude

$$\|g\|_{L^2(0,1)} = \left(\sum_{j=1}^{\infty} |\widehat{g}_j|^2 \right)^{1/2} = \|Ug\|_{H^2(\mathbb{T}^\infty)}, \tag{3.17}$$

see also [20] for more details.

The mechanism of transforming the space $L^2(0,1)$ into the Hardy space $H^2(\mathbb{T}^\infty)$ together with the last identity between their norms confirm the existence of an isometry isomorphism $U : L^2(0,1) \longrightarrow H^2(\mathbb{T}^\infty)$.

Now, associate to g in (3.16) the function

$$\begin{aligned} Mg(x) &= \sum_{n \in \mathcal{F}} \widehat{f}_n M_n \left(\sum_{j=1}^{\infty} \widehat{g}_j e_j(x) \right) = \sum_{j=1}^{\infty} \widehat{g}_j \sum_{n \in \mathcal{F}} \widehat{f}_n e_{nj}(x) \\ &= \sum_{\mu \in \mathbb{N}_0^\infty} \widehat{g}_j \sum_{n \in \mathcal{F}} \widehat{f}_n M_{\mathfrak{p}_1}^{\nu_{\mathfrak{p}_1}(n)} \cdots M_{\mathfrak{p}_d}^{\nu_{\mathfrak{p}_d}(n)} M_{\mathfrak{q}_1}^{\mu_{\mathfrak{q}_1}(j)} \cdots M_{\mathfrak{q}_s}^{\mu_{\mathfrak{q}_s}(j)} e_1(x). \end{aligned}$$

It is evident that, $Mg \in L^2(0,1)$, i.e.

$$\begin{aligned}
\|Mg\|_{L^2(0,1)} &= \left\| \sum_{n\in\mathcal{F}} \widehat{f}_n M_n \left(\sum_{j=1}^{\infty} \widehat{g}_j e_j(x) \right) \right\|_{L^2(0,1)} \\
&\leq \sum_{n\in\mathcal{F}} |\widehat{f}_n| \left\| M_n \left(\sum_{j=1}^{\infty} \widehat{g}_j e_j(x) \right) \right\|_{L^2(0,1)} \\
&= \sum_{n\in\mathcal{F}} |\widehat{f}_n| \left\| \sum_{j=1}^{\infty} \widehat{g}_j e_j(x) \right\|_{L^2(0,1)} \\
&= \sum_{n\in\mathcal{F}} |\widehat{f}_n| \left(\sum_{j=1}^{\infty} |\widehat{g}_j|^2 \right)^{1/2} < \infty.
\end{aligned}$$

This is due to the triangle inequality, identity (3.17) and Lemma 3.1.

When the operator U is applied to Mg, we arrive at the identity

$$U\left(Mg\right)(z) = m(z)Ug(z), \qquad z \in \mathbb{T}^{\infty}. \tag{3.18}$$

This powerful connection between the operator M and the symbol (multiplier) m was first established in [32]. Then a more general framework was developed in the context of Hardy-Dirichlet spaces in [20].

The statement (3.18) has the interpretation that replacing e_j with e_{nj} for $n \in \mathcal{F}$ and $j \in \mathbb{N}$ corresponds to multiplication by $m(z)$ on the U-transformed side. See Figure 3.1 for further clarification observing that,

$$M = U^{-1}\mathscr{M}_m U. \tag{3.19}$$

Theorem 3.2.

$$\|M\|_{\mathcal{B}(L^2(0,1))} = \|m\|_{\infty}.$$

Proof. According to the statements (3.15) and (3.19) and since U is an invertible isometry we have,

$$\begin{aligned}
\|M\|_{\mathcal{B}(L^2(0,1))} &= \|UM\|_{L^2(0,1)\to H^2(\mathbb{T}^\infty)} \\
&= \|\mathscr{M}_m U\|_{L^2(0,1)\to H^2(\mathbb{T}^\infty)} = \|m\|_{\infty}.
\end{aligned}$$

$$\begin{tikzcd} L^2(0,1) \arrow[r, "M"] \arrow[d, "U"'] & L^2(0,1) \\ H^2(\mathbb{T}^\infty) \arrow[r, "\mathscr{M}_m"'] & H^2(\mathbb{T}^\infty) \arrow[u, "U^{-1}"'] \end{tikzcd}$$

Figure 3.1: Description of the intertwine between M and $\mathscr{M}_m$ by means of the isometry U.

The result follows. $\square$

Consider M given by (3.12). The result from the following theorem will be invoked below in several places.

Theorem 3.3. *If $m(z) \neq 0$ on the closed polydisc $\overline{\mathbb{D}}^d$, then the linear operator M is a homeomorphism. Moreover,*

$$\|M^{-1}\|^{-1}_{\mathcal{B}(L^2(0,1))} = \inf_{z \in \mathbb{T}^d} |m(z)|.$$

Proof. From (3.13), observe that $m \in H^\infty(\mathbb{T}^d)$. According to Fatou's Theorem 2.9 and Remark 2.1, the function $m(z)$ can be extended by analytic continuity to the polydisc $\mathbb{D}^d$. Since the function $m(z)$ is analytic and bounded away from zero on $\overline{\mathbb{D}}^d$, then $1/m(z)$ is also analytic on $\overline{\mathbb{D}}^d$. Moreover the infimum of $m(z)$ lies on the distinguished boundary $\mathbb{T}^d$ (Minimum Modulus Principle). According to the Expansion Theorem (Taylor) [24, Theorem 1, pp. 79], the function $1/m(z)$ has a unique Maclaurin expansion in $z = (z_1, \cdots, z_d) \in \overline{\mathbb{D}}^d$ with derivatives given in terms of the Cauchy Integral Formula and this series converges to $1/m(z)$ in $\overline{\mathbb{D}}^d$. Now, replacing z_j by $M_{\mathfrak{p}_j}$ gives a series converging in norm to M^{-1} (see also [23, Chapter I, sections 4.4, 4.5, pp. 29 & 1.7, pp. 8] for further details). The first statement follows.

According to this assertion and statement (3.19), we arrive at the following diagram:

$$\begin{tikzcd} L^2(0,1) & L^2(0,1) \arrow[l, "M^{-1}"'] \arrow[d, "U"] \\ H^2(\mathbb{T}^\infty) \arrow[u, "U^{-1}"] & H^2(\mathbb{T}^\infty) \arrow[l, "\mathscr{M}_m^{-1}"] \end{tikzcd}$$

Notice that $\mathscr{M}_m^{-1} = \mathscr{M}_{m^{-1}}$ and is defined by $\mathscr{M}_m^{-1}F(z) = m^{-1}(z)F(z), F \in H^2(\mathbb{T}^\infty)$ (see the proof of Theorem 7.4 for $r = 2$). Moreover,

$$M^{-1} = U^{-1}\mathscr{M}_m^{-1}U.$$

Hence,

$$\begin{aligned} \|M^{-1}\|_{\mathcal{B}(L^2(0,1))} &= \|\mathscr{M}_m^{-1}\|_{\mathcal{B}(H^2(\mathbb{T}^\infty))} \\ &= \sup_{z\in\mathbb{T}^d} |m^{-1}(z)| = \left(\inf_{z\in\mathbb{T}^d} |m(z)|\right)^{-1}. \end{aligned}$$

□

Corollary 3.2. *Assume that the Fourier coefficients of f are such that*

$$|\widehat{f}_j| \leq \phi_j, \qquad \forall j \in \mathbb{N} \tag{3.20}$$

for a sequence $\{\phi_j\}_{j=1}^\infty \in \ell^1(\mathbb{N})$. Set $\Phi = \sum_{j=1}^\infty \phi_j$. Let $k, d \in \mathbb{N}$ be fixed. Let

$$\mathcal{F} = \{1,\, \mathfrak{p}_1,\, \mathfrak{p}_1^2,\, \ldots,\, \mathfrak{p}_d,\, \mathfrak{p}_d^2\}.$$

Set

$$m(z) = \widehat{f}_1 + \sum_{j=1}^d \left(\widehat{f}_{\mathfrak{p}_j} z_j + \widehat{f}_{\mathfrak{p}_j^2} z_j^2\right), \qquad \forall z \in \mathbb{D}^d$$

and

$$\omega = \inf\{|m(z)| : z \in \mathbb{T}^d\}.$$

If

$$\sum_{j\in\mathcal{F}\setminus\{1\}} |\widehat{f}_j| < |\widehat{f}_1| \tag{3.21}$$

and

$$\omega - \Phi + \sum_{j\in\mathcal{F}} |\widehat{f}_j| + \sum_{j=1}^{k} \left(\phi_j - |\widehat{f}_j|\right) > 0, \tag{3.22}$$

then $\mathfrak{E}_f$ *is a Riesz basis of* $L^2(0,1)$.

Proof. Due to (3.21), $m(z)$ has no zeros on $\overline{\mathbb{D}}^d$ and so $\omega > 0$. According to Theorem 3.3, M is invertible and $\|M^{-1}\|^{-1} = \omega$. Now

$$\begin{aligned}
\|V\| &\leq \sum_{j\in\mathbb{N}\setminus\mathcal{F}} |\widehat{f}_j| = \sum_{j\in\mathbb{N}} |\widehat{f}_j| - \sum_{j\in\mathcal{F}} |\widehat{f}_j| \\
&\leq \sum_{j=1}^{k} |\widehat{f}_j| + \sum_{j=k+1}^{\infty} \phi_j - \sum_{j\in\mathcal{F}} |\widehat{f}_j| \\
&= \sum_{j=1}^{k} \left(|\widehat{f}_j| - \phi_j\right) + \Phi - \sum_{j\in\mathcal{F}} |\widehat{f}_j|.
\end{aligned}$$

Statement (3.22) implies $\|V\| < \omega$. Then due to (3.6), the operator T is invertible. This in turns ensures that $\mathfrak{E}_f$ is a Riesz basis of $L^2(0,1)$. $\square$

The following more refined natural extension of [7, Corollaries 4.3 and 4.4] can also be deduced from Theorems 3.2 and 3.3 for $d = 1$ and $m(z) \neq 0$ on the closed unit disc $\overline{\mathbb{D}}$. The mechanism developed in this chapter associates the operator $M = \sum_{j=0}^{\infty} \widehat{f}_{\mathfrak{p}^j} M_{\mathfrak{p}^j}$ to the multiplication operator $\mathscr{M}_m : H^2(\mathbb{D}) \longrightarrow H^2(\mathbb{D})$ where $m(z) = \sum_{j=0}^{\infty} \widehat{f}_{\mathfrak{p}^j} z^j$. This association is allowed via the isometry isomorphism $U : L^2(0,1) \longrightarrow H^2(\mathbb{D})$ such that (3.19) holds. Moreover according to Theorems 7.1 and 7.2, $\|\mathscr{M}_m\|_{\mathcal{B}(H^2(\mathbb{D}))} = \sup_{z\in\mathbb{T}} |m(z)|$ and $\|\mathscr{M}_m^{-1}\|^{-1}_{\mathcal{B}(H^2(\mathbb{D}))} = \inf_{z\in\mathbb{T}} |m(z)|$. See [7, Theorem 3.2] for further details.

Corollary 3.3. *Let* $d = 1$. *Assume that*

$$0 < \widehat{f}_{\mathfrak{p}^2} < \widehat{f}_1 \quad \text{and} \quad \widehat{f}_{\mathfrak{p}^2} + \widehat{f}_1 > \pm|\widehat{f}_{\mathfrak{p}}|. \tag{3.23}$$

Either of the following two conditions ensure that $\mathfrak{E}_f$ *is a Riesz basis of* $L^2(0,1)$.

(a) $|\widehat{f}_{\mathfrak{p}}(\widehat{f}_{\mathfrak{p}^2} + \widehat{f}_1)| \geq |4\widehat{f}_{\mathfrak{p}^2}\widehat{f}_1|$ *and*

$$\sum_{j\in\mathbb{N}\setminus\{1,\mathfrak{p}^2\}} |\widehat{f}_j| < \widehat{f}_1 + \widehat{f}_{\mathfrak{p}^2}, \tag{3.24}$$

(b) $|\widehat{f}_{\mathfrak{p}}(\widehat{f}_{\mathfrak{p}^2}+\widehat{f}_1)| < |4\widehat{f}_{\mathfrak{p}^2}\widehat{f}_1|$ *and*

$$\sum_{j\in\mathbb{N}\setminus\{1,\mathfrak{p},\mathfrak{p}^2\}} |\widehat{f}_j| < (\widehat{f}_1-\widehat{f}_{\mathfrak{p}^2})\sqrt{1-\frac{\widehat{f}_{\mathfrak{p}}^2}{4\widehat{f}_1\widehat{f}_{\mathfrak{p}^2}}}. \tag{3.25}$$

Proof. Let

$$m(z)=\widehat{f}_1+\widehat{f}_{\mathfrak{p}}z+\widehat{f}_{\mathfrak{p}^2}z^2 \qquad z\in\mathbb{D}.$$

According to the calculations performed in the proof of [7, Lemma 4.1] for the case $\mathfrak{p}=3$, if (3.23) holds true, then $m(z)$ has its zeros outside $\overline{\mathbb{D}}$. Moreover,

$$|\widehat{f}_{\mathfrak{p}}(\widehat{f}_{\mathfrak{p}^2}+\widehat{f}_1)| \geq |4\widehat{f}_{\mathfrak{p}^2}\widehat{f}_1| \quad\Rightarrow\quad \omega=\inf_{z\in\mathbb{T}}|m(z)|=\widehat{f}_1+\widehat{f}_{\mathfrak{p}^2}-|\widehat{f}_{\mathfrak{p}}|$$

and

$$|\widehat{f}_{\mathfrak{p}}(\widehat{f}_{\mathfrak{p}^2}+\widehat{f}_1)| < |4\widehat{f}_{\mathfrak{p}^2}\widehat{f}_1| \quad\Rightarrow\quad \omega=\inf_{z\in\mathbb{T}}|m(z)|=(\widehat{f}_1-\widehat{f}_{\mathfrak{p}^2})\sqrt{1-\frac{\widehat{f}_{\mathfrak{p}}^2}{4\widehat{f}_1\widehat{f}_{\mathfrak{p}^2}}}.$$

For more details, see the argument in [7, Theorem 3.2, Lemma 4.1]. □

Remark 3.2. *In Chapters 4, 5 and 6, we will consider the construction of the isometries M_k with g^* defined by means of even extensions and cosine Fourier coefficients rather than sine Fourier coefficients. Theorem 3.1 and Corollaries 3.1, 3.2 and 3.3 have identical analogues in this case. See Section 4.3 for further details.*

Chapter 4

Schauder Basis and Regularity Properties of p-trigonometric Functions

In this chapter we are going to address the question proposed in Section 1.2 for two particular choices $f(x) = \cos_p(\pi_p x)$ and $f(x) = \sin_p(\pi_p x)$. We set the *one-term* (Theorem 3.1) and the improved *one-term* (Corollary 3.1) criteria for determining invertibility of the change of coordinates map between e_j and f_j functions.

The study of the former of these functions constitutes the major component in this chapter for which the proof is divided into the cases $p \in (1, 2)$ and $p \in (2, \infty)$. Here we collect various properties of the generalised cosine functions which will be useful in studying bases properties. We then establish precise upper bounds on the asymptotic behaviour of Fourier coefficients in the two cases $p \in (1, 2)$ and $p \in (2, \infty)$. We find two thresholds, $p_0 < 2$ and $p_1 > 2$, by means of analytical calculations which are then improved via some numerical approximations to $\hat{p}_{0,1} < p_0$ and $\hat{p}_{1,1} > p_1$ respectively. The family $\mathfrak{E}_{\cos_p}$ forms a Schauder basis of $L^r(0, 1)$ for $r \in (1, \infty)$ and all $p \in (\hat{p}_{0,1}, \hat{p}_{1,1})$ (see Table 1.1).

Previous investigations concerning bases properties of the family $\mathfrak{E}_{\sin_p}$ were settled at the threshold $\tilde{p}_1$ such that the set is a Schauder basis in $L^r(0, 1)$ for all $r \in (1, \infty)$ and $p \in [\tilde{p}_1, \infty)$ (see [7] and Table 1.1 for more details). This was obtained using Theorem 3.1. The last section of this chapter employs Corollary 3.1 and shows improvements in the threshold of p upon those achieved previously. It also examines the regularity of these functions by means of the upper bound estimates of the Fourier coefficients.

Throughout this and the following chapters we will consider the definitions, the properties and the notations of previous sections when studying the generalised trigonometric functions.

4.1 Properties of generalised cosine functions

Lemma 4.1. *For all* $x \in [0, \frac{1}{2})$,

(a)

$$\cos_p(\pi_p x) = \left[\sin_{p'}\left(\pi_{p'}\left(\frac{1}{2} - x\right)\right)\right]^{p'-1}$$

(b)

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos_p(x) = -(\sin_p x)^{p-1}(\cos_p x)^{2-p}$$

(c)

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\cos_p(x) = (\sin_p x)^{p-2}(\cos_p x)^{3-2p}[2 - p - \cos_p^p x].$$

Proof. The calculations leading to *(a)* and *(b)* can be found in the proofs of [9, Proposition 2.2] and [9, Proposition 2.1], respectively. From (1.8) and *(b)* we get

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\cos_p x &= (2-p)(\sin_p x)^{2p-2}(\cos_p x)^{3-2p} - (p-1)(\sin_p x)^{p-2}(\cos_p x)^{3-p} \\ &= (\sin_p x)^{p-2}(\cos_p x)^{3-2p}\left[(2-p)\sin_p^p x - (p-1)\cos_p^p x\right],\end{aligned}$$

which is *(c)*. $\square$

The following inequality will be important below.

Lemma 4.2. *Let* $1 < p_1 \leq p_2 < \infty$ *and* $x \in [0, \frac{1}{2}]$. *Then*

$$\cos_{p_1}(\pi_{p_1} x) \leq \cos_{p_2}(\pi_{p_2} x). \tag{4.1}$$

Proof. A direct evaluation at $x = 0$ and $x = 1/2$ gives equality for all p_1 and p_2 at these points, so these two cases are immediate. Let $x \in (0, \frac{1}{2})$ be fixed. Since p' is decreasing in $p \in (1, \infty)$, from Lemma 1.7*(c)* it follows that

$$\frac{\mathrm{d}}{\mathrm{d}p} \sin_{p'} \Big(\pi_{p'}\Big(\frac{1}{2} - x\Big)\Big) \geq 0 \qquad \forall p \in (1, \infty).$$

Note that, $0 < \sin_{p'}(\pi_{p'}(\frac{1}{2} - x)) < 1$ and hence $\ln(\sin_{p'}(\pi_{p'}(\frac{1}{2} - x))) < 0$. Substituting the identity from Lemma 4.1*(a)*, yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}p} \cos_p(\pi_p x) = \frac{\mathrm{d}}{\mathrm{d}p} \Big[\sin_{p'} \Big(\pi_{p'}\Big(\frac{1}{2} - x\Big)\Big)\Big]^{\frac{1}{p-1}} \\
&= \left[-\frac{\ln(\sin_{p'}(\pi_{p'}(\frac{1}{2} - x)))}{(p-1)^2} + \frac{\frac{\mathrm{d}}{\mathrm{d}p}\left[\sin_{p'}(\pi_{p'}(\frac{1}{2} - x))\right]}{(p-1)\sin_{p'}(\pi_{p'}(\frac{1}{2} - x))} \right] \cos_p(\pi_p x) > 0.
\end{aligned}$$

The result follows. □

4.1.1 The case $p \in (1, 2)$

For $p \in (1, 2)$, let $\mathrm{u}_p : [0, \frac{1}{2}] \longrightarrow \mathbb{R}$ be given by

$$\begin{aligned}
\mathrm{u}_p(x) :&= \frac{\mathrm{d}}{\mathrm{d}y} \cos_p y \Big|_{y = \pi_p x} \\
&= -(\sin_p(\pi_p x))^{p-1} (\cos_p(\pi_p x))^{2-p}.
\end{aligned}$$

This function will simplify the notation when we determine estimates for the Fourier coefficients of the p-cosine functions in Section 4.2.1. Here and everywhere below we write

$$\mathcal{M}_p := (p-1)^{\frac{p-1}{p}} (2-p)^{\frac{2-p}{p}}. \tag{4.2}$$

Lemma 4.3. *Let $p \in (1, 2)$. Then*

(a) $\mathrm{u}_p(x) \leq 0$ *for all* $x \in [0, \frac{1}{2}]$

(b) $\mathrm{u}_p(x) = 0$ *if and only if* $x = 0$ *or* $x = \frac{1}{2}$

(c) $\mathrm{u}_p(x) = -\mathcal{M}_p$ *for* $x \in [0, \frac{1}{2}]$ *if and only if* $x = m_p \in (0, \frac{1}{2})$*, where* m_p *is the unique point such that* $\cos_p^p(\pi_p m_p) = 2 - p$

(d) $\mathrm{u}_p : [0, m_p] \longrightarrow [-\mathcal{M}_p, 0]$ *is decreasing*

(e) $\mathrm{u}_p : [m_p, \frac{1}{2}] \longrightarrow [-\mathcal{M}_p, 0]$ *is increasing*

(f) $\min_{x \in [0, \frac{1}{2}]} \mathrm{u}_p(x) = -\mathcal{M}_p$.

Proof. Since $\sin_p(\pi_p x)$ and $\cos_p(\pi_p x)$ are non-negative over $[0, \frac{1}{2}]$, then *(a)* holds true. Since $\sin_p(\pi_p x)$ only vanishes at $x = 0$ and $\cos_p(\pi_p x)$ only vanishes at $x = \frac{1}{2}$ in this interval, then *(b)* holds true.

Lemma 4.1*(c)* gives

$$\mathrm{u}_p'(x) = \pi_p (\sin_p(\pi_p x))^{p-2} (\cos_p(\pi_p x))^{3-2p} [2 - p - \cos_p^p(\pi_p x)].$$

Neither $\sin_p(\pi_p x)$ nor $\cos_p(\pi_p x)$ vanish in $(0, \frac{1}{2})$. On the other hand, $\cos_p^p(0) = 1 > 2-p$, $\cos_p^p(\frac{\pi_p}{2}) = 0 < 2-p$ and $\cos_p^p(\pi_p x)$ is decreasing for $x \in (0, \frac{1}{2})$. Then according to the Intermediate Value Theorem the term $\cos_p^p(\pi_p x) + p - 2$ indeed vanishes at the unique point $m_p \in (0, \frac{1}{2})$ as stated in *(c)*.

At m_p,

$$\begin{aligned} \mathrm{u}_p(m_p) &= -(\sin_p(\pi_p m_p))^{p-1} (\cos_p(\pi_p m_p))^{2-p} \\ &= -(1 - \cos_p^p(\pi_p m_p))^{\frac{p-1}{p}} (\cos_p(\pi_p m_p))^{2-p} = -\mathcal{M}_p. \end{aligned}$$

Hence, the proof of *(d)* and *(e)*, and thus of *(f)*, is achieved as follows. Just observe that in the expression for $\mathrm{u}_p'(x)$ above, $\cos_p^p(\pi_p x) > 2 - p$ for $x \in [0, m_p)$ and $\cos_p^p(\pi_p x) < 2 - p$ for $x \in (m_p, \frac{1}{2})$, because $\cos_p(\pi_p x)$ is decreasing in $x \in (0, \frac{1}{2})$. $\square$

According to parts *(d)* and *(e)* of Lemma 4.3, the function u_p is invertible, when restricted to the segments $[0, m_p]$ and $[m_p, \frac{1}{2}]$. We denote the inverses by $\mathrm{w}_{1,p} : [-\mathcal{M}_p, 0] \longrightarrow [0, m_p]$ and $\mathrm{w}_{2,p} : [-\mathcal{M}_p, 0] \longrightarrow [m_p, \frac{1}{2}]$, respectively, so that

$$\mathrm{u}_p(\mathrm{w}_{k,p}(x)) = x \qquad \forall x \in [-\mathcal{M}_p, 0] \quad k = 1, 2.$$

4.1.2 The case $p \in (2,\infty)$

For $p \in (2,\infty)$, let $\mathrm{v}_p : (0,\frac{1}{2}] \longrightarrow [0,\infty)$ be given by

$$\mathrm{v}_p(x) := (p'-1)(\sin_{p'}(\pi_{p'}x))^{p'-2}\cos_{p'}(\pi_{p'}x).$$

Let us summarise various properties of this function, which will be employed in Section 4.2.2.

Lemma 4.4. *Let $p \in (2,\infty)$. Then*

(a) v_p *is decreasing in* $(0,\frac{1}{2}]$

(b) $\lim_{x\to 0^+} x\,\mathrm{v}_p(x) = 0$

(c) $\lim_{x\to 0^+} \mathrm{v}_p(x) = +\infty$ *and* $\mathrm{v}_p(\frac{1}{2}) = 0$

(d) $\lim_{x\to 0^+} \mathrm{v}'_p(x) = -\infty$ *and* $\mathrm{v}'_p(\frac{1}{2}) = 0.$

Proof. For $p \in (2,\infty)$, $p' \in (1,2)$ and so $p'-2<0$. Since, $\sin_{p'}(\pi_{p'}x)$ is increasing and $\cos_{p'}(\pi_{p'}x)$ is decreasing in $x \in (0,\frac{1}{2})$, then *(a)* holds true.

Let us show *(b)*. L'Hôpital's Rule gives

$$\lim_{x\to 0^+}\frac{x}{(\sin_{p'}(\pi_{p'}x))^{2-p'}} = \lim_{x\to 0^+}\frac{(\sin_{p'}(\pi_{p'}x))^{p'-1}}{(2-p')\pi_{p'}\cos_{p'}(\pi_{p'}x)} = 0.$$

Then,

$$\lim_{x\to 0^+} x\,\mathrm{v}_p(x) = \lim_{x\to 0^+}(p'-1)\frac{x\cos_{p'}(\pi_{p'}x)}{(\sin_{p'}(\pi_{p'}x))^{2-p'}} = 0,$$

as claimed in *(b)*.

Both statements *(c)* and *(d)* follow directly from (1.8), the expression

$$\mathrm{v}'_p(x) = (p'-1)\pi_{p'}(\sin_{p'}(\pi_{p'}x))^{p'-3}(\cos_{p'}(\pi_{p'}x))^{2-p'}\left[(p'-1)\cos_{p'}^{p'}(\pi_{p'}x)-1\right],$$

and continuity of $\sin_p$ and $\cos_p$ at $x=0$. □

According to this lemma, there exists a function $\mathrm{z}_p : [0,\infty) \to (0,\frac{1}{2}]$ such that z_p is the inverse function of v_p. This inverse function has the following characteristics.

(a) z_p is decreasing in $[0,\infty)$

(b) $\mathrm{z}_p(0)=\frac{1}{2}$ and $\lim\limits_{x\to\infty}\mathrm{z}_p(x)=0$

(c) $\lim\limits_{x\to 0^+}\mathrm{z}_p'(x)=-\infty$ and $\lim\limits_{x\to\infty}\mathrm{z}_p'(x)=0$.

4.2 The Fourier coefficients of the p-cosine functions

Let $a_j(p)$ and $b_j(p)$ be the sine and cosine Fourier coefficients of $\sin_p(\pi_p x)$ and $\cos_p(\pi_p x)$, respectively. Since $\sin_p$ is even and $\cos_p$ is odd around $\frac{1}{2}$, $a_j(p)=b_j(p)=0$ for all $j\equiv_2 0$ and $p\in(1,\infty)$. Then

$$a_j := a_j(p) = \begin{cases} 0 & j\equiv_2 0 \\ 2\int_0^1 \sin_p(\pi_p x)\sin(j\pi x)\mathrm{d}x & j\equiv_2 1. \end{cases} \tag{4.3}$$

and

$$b_j := b_j(p) = \begin{cases} 0 & j\equiv_2 0 \\ 2\int_0^1 \cos_p(\pi_p x)\cos(j\pi x)\mathrm{d}x & j\equiv_2 1. \end{cases} \tag{4.4}$$

Lemma 4.5. *For $j\in\mathbb{N}$,*

$$b_j(p)=\frac{j\pi}{\pi_p}a_j(p).$$

Proof. Let $j\equiv_2 1$. Integration by parts alongside with the fact that $\cos_p(\pi_p x)$ and $\cos(j\pi x)$ are odd with respect to $\frac{1}{2}$, yield

$$\begin{aligned} b_j(p) &= 2\int_0^1 \cos_p(\pi_p x)\cos(j\pi x)\mathrm{d}x = 4\int_0^{\frac{1}{2}} \cos_p(\pi_p x)\cos(j\pi x)\mathrm{d}x \\ &= \frac{4}{\pi_p}\cos(j\pi x)\sin_p(\pi_p x)\Big|_0^{\frac{1}{2}} + \frac{4j\pi}{\pi_p}\int_0^{\frac{1}{2}}\sin_p(\pi_p x)\sin(j\pi x)\mathrm{d}x \\ &= \frac{j\pi}{\pi_p}a_j(p). \end{aligned}$$

□

We now find estimates on $|b_j(p)|$ in terms of the parameter $p \in (1, \infty)$.

4.2.1 The case $p \in (1, 2)$

Lemma 4.6. *For $p \in (1,2)$, let $\mathcal{M}_p > 0$ be given by (6.3). Then*

$$|b_j(p)| < \frac{8\pi_p}{j^2\pi^2}\mathcal{M}_p \qquad \forall j \geq 1.$$

Proof. Integrate by parts twice to get

$$\begin{aligned}
b_j(p) &= 4\int_0^{\frac{1}{2}} \cos_p(\pi_p x)\cos(j\pi x)\mathrm{d}x \\
&= \frac{4}{j\pi}\cos_p(\pi_p x)\sin(j\pi x)\Big|_0^{\frac{1}{2}} - \frac{4\pi_p}{j\pi}\int_0^{\frac{1}{2}} \mathrm{u}_p(x)\sin(j\pi x)\mathrm{d}x \\
&= -\frac{4\pi_p}{j\pi}\int_0^{\frac{1}{2}} \mathrm{u}_p(x)\sin(j\pi x)\mathrm{d}x \\
&= \frac{4\pi_p}{j^2\pi^2}\mathrm{u}_p(x)\cos(j\pi x)\Big|_0^{\frac{1}{2}} - \frac{4\pi_p}{j^2\pi^2}\int_0^{\frac{1}{2}} \mathrm{u}_p'(x)\cos(j\pi x)\mathrm{d}x.
\end{aligned}$$

From the identities in Lemma 4.3*(b)*, it follows that the boundary term in the fourth equality always vanishes. Thus,

$$\begin{aligned}
b_j(p) &= -\frac{4\pi_p}{j^2\pi^2}\int_0^{\frac{1}{2}} \mathrm{u}_p'(x)\cos(j\pi x)\mathrm{d}x \\
&= -\frac{4\pi_p}{j^2\pi^2}\Big(\int_0^{m_p} \mathrm{u}_p'(x)\cos(j\pi x)\mathrm{d}x + \int_{m_p}^{\frac{1}{2}} \mathrm{u}_p'(x)\cos(j\pi x)\mathrm{d}x\Big) \\
&= -\frac{4\pi_p}{j^2\pi^2}\Big(\int_0^{-\mathcal{M}_p} \cos(j\pi\,\mathrm{w}_{1,p}(s))ds + \int_{-\mathcal{M}_p}^{0} \cos(j\pi\,\mathrm{w}_{2,p}(s))\mathrm{d}s\Big).
\end{aligned}$$

Hence,

$$\begin{aligned}
|b_j(p)| &\leq \frac{4\pi_p}{j^2\pi^2}\Big[\int_{-\mathcal{M}_p}^{0} |\cos(j\pi\,\mathrm{w}_{1,p}(s))|ds + \int_{-\mathcal{M}_p}^{0} |\cos(j\pi\,\mathrm{w}_{2,p}(s))|\mathrm{d}s\Big] \\
&< \frac{8\pi_p}{j^2\pi^2}\mathcal{M}_p.
\end{aligned}$$

The result follows. □

4.2.2 The case $p \in (2, \infty)$

Let $p \in (2, \infty)$. According to Lemma 4.1*(a)*,

$$b_j(p) = 4 \int_0^{\frac{1}{2}} \left[\sin_{p'} \left(\pi_{p'} \left(\frac{1}{2} - x \right) \right) \right]^{\frac{1}{p-1}} \cos(j\pi x) \mathrm{d}x.$$

Since $\cos(j\pi(\frac{1}{2} - t)) = (-1)^{\frac{j-1}{2}} \sin(j\pi t)$ for $j \equiv_2 1$, changing variables to $t = \frac{1}{2} - x$ gives

$$b_j(p) = (-1)^{\frac{j-1}{2}} 4 \int_0^{\frac{1}{2}} (\sin_{p'}(\pi_{p'} t))^{\frac{1}{p-1}} \sin(j\pi t) \mathrm{d}t.$$

By virtue of Lemma 4.4 and integration by parts twice, then

$$\begin{aligned} b_j(p) &= (-1)^{\frac{j-1}{2}} \frac{4\pi_{p'}}{j\pi} \int_0^{\frac{1}{2}} \mathrm{v}_p(t) \cos(j\pi t) \mathrm{d}t \\ &= (-1)^{\frac{j-1}{2}} \frac{4\pi_{p'}}{j\pi} \left[\frac{1}{j\pi} \mathrm{v}_p(t) \sin(j\pi t) \Big|_0^{\frac{1}{2}} - \frac{1}{j\pi} \int_0^{\frac{1}{2}} \mathrm{v}_p'(t) \sin(j\pi t) \mathrm{d}t \right] \\ &= (-1)^{\frac{j+1}{2}} \frac{4\pi_{p'}}{j^2\pi^2} \int_0^{\frac{1}{2}} \mathrm{v}_p'(t) \sin(j\pi t) \mathrm{d}t \\ &= (-1)^{\frac{j+3}{2}} \frac{4\pi_{p'}}{j^2\pi^2} \int_0^{\infty} \sin\left(j\pi \, \mathrm{z}_p(y)\right) \mathrm{d}y. \end{aligned} \tag{4.5}$$

Lemma 4.7. *Let $p \in (2, \infty)$. Then*

$$|b_j(p)| < \frac{2\pi_{p'}}{\pi^2(p-1)} \left[2 + \frac{\pi^2}{2}(p-2) \right] j^{-p'}, \qquad \forall j \geq 3.$$

Proof. Since $p \in (2, \infty)$, then $p' \in (1, 2)$. Let $r = p' - 1$. In view of Lemma 1.7*(c)*, we have

$$\mathrm{v}_p(t) < r \left[\sin_{p'}(\pi_{p'} t)\right]^{r-1} < r \big[\sin(\pi t)\big]^{r-1}$$

and so

$$\mathrm{z}_p(y) < \frac{1}{\pi} \arcsin \left[\left(\frac{y}{r} \right)^{\frac{1}{r-1}} \right] =: \mathrm{r}_p(y), \qquad \forall y \in [r, \infty). \tag{4.6}$$

Set

$$\eta(j) := r \left[\sin \left(\frac{\pi}{2j} \right) \right]^{r-1}.$$

Then,

$$\mathrm{r}_p(\eta(j)) = \frac{1}{2j} < \frac{1}{2}.$$

Here we use the requirement $j \geq 3$, in order to make sure that the arc-sine does not change branches.

Set

$$\mathrm{J}_1 = \int_0^{\eta(j)} \mathrm{d}x = \eta(j)$$

and

$$\mathrm{J}_2 = \int_{\eta(j)}^{\infty} \sin\left(j\pi\, \mathrm{r}_p(y)\right) \mathrm{d}y.$$

Since $0 < j\pi\, \mathrm{z}_p(y) \leq j\pi\, \mathrm{z}_p(\eta(j)) < j\pi\, \mathrm{r}_p(\eta(j)) = \frac{\pi}{2}$, then $0 < \sin(j\pi\, \mathrm{z}_p(y)) < \sin(j\pi\, \mathrm{r}_p(y))$ for $y \in [\eta(j), \infty)$. Then, (4.5) yields

$$|b_j(p)| < \frac{4\pi_{p'}}{j^2\pi^2}(\mathrm{J}_1 + \mathrm{J}_2).$$

Let us estimate an upper bound for J_2. Changing variables to

$$t = j\pi\, \mathrm{r}_p(y) \Longleftrightarrow y = r\left[\sin\left(\frac{t}{j}\right)\right]^{r-1}$$

gives

$$\begin{aligned}
\mathrm{J}_2 &= \int_0^{\frac{\pi}{2}} \frac{r(1-r)}{j}\left[\sin\left(\frac{t}{j}\right)\right]^{r-2} \cos\left(\frac{t}{j}\right) \sin(t)\mathrm{d}t \\
&= r(1-r)\int_0^{\frac{\pi}{2}} \left[\sin\left(\frac{t}{j}\right)\right]^{r-1} \left[\frac{\frac{t}{j}}{\sin\left(\frac{t}{j}\right)}\right]\left(\frac{\sin t}{t}\right) \cos\left(\frac{t}{j}\right) \mathrm{d}t.
\end{aligned}$$

Note that,

$$\sup_{0<\theta\leq\frac{\pi}{2}} \frac{\theta}{\sin\theta} = \frac{\pi}{2}, \qquad \sup_{0<\theta\leq\frac{\pi}{2}} \frac{\sin\theta}{\theta} = 1 \tag{4.7}$$

and

$$0 < t < j\pi\, \mathrm{r}_p(\eta(j)) = \frac{\pi}{2}.$$

Here we are using once again the fact that $j \geq 3$. Then

$$\mathrm{J}_2 \leq \frac{\pi}{2} r(1-r) \int_0^{\frac{\pi}{2}} \left[\sin\left(\frac{t}{j}\right)\right]^{r-1} \cos\left(\frac{t}{j}\right) \mathrm{d}t.$$

Changing variables to

$$\tau = \sin\left(\frac{t}{j}\right),$$

yields

$$\mathrm{J}_2 \leq \frac{j\pi}{2} r(1-r) \int_0^{\sin\frac{\pi}{2j}} \tau^{r-1} \mathrm{d}\tau = \frac{j\pi}{2}(1-r)\left[\sin\left(\frac{\pi}{2j}\right)\right]^r.$$

Then

$$|b_j| < \frac{2\pi_{p'}}{j^2\pi^2}\left[2 + \frac{j\pi(1-r)}{r}\sin\left(\frac{\pi}{2j}\right)\right]\eta(j).$$

According to (4.7), we get

$$\eta(j) \leq r j^{1-r}$$

and

$$|b_j(p)| < \frac{2\pi_{p'} r}{j^2\pi^2}\left[2 + \frac{j\pi(1-r)}{r}\frac{\pi}{2j}\right] j^{1-r}. \tag{4.8}$$

Simplifying the expression on the right hand side, ensures the validity of the lemma.

□

Proposition 4.1. *For all $p \in (1, \infty)$,*

$$\sum_{j=1}^{\infty} |b_j(p)| < \infty.$$

Proof. This is a direct consequence of Lemmas 4.6 and 4.7. See (4.13) and (4.23) below. □

4.3 The change of coordinates map

We now derive various properties of the change of coordinates maps that take the 2-cosine functions into the p-cosine functions. The same argument as in Section 3.1 can be applied now to the case when an even extension to $\mathbb{R}$ is required (see, Remark 3.2).

Most of the material in this section can also be found in [5], [9], [12] and [7].

Given any $g \in L^r(0,1)$, define

$$M_n g(x) = g^*(nx), \tag{4.9}$$

where g^* is the 2-periodic even extension of g to the whole real line. According to Lemma 3.1, M_n is a bounded linear isometry from $L^r(0,1)$ to itself.

Let $e_n(x) = \cos(n\pi x)$. Let $r \in (1,\infty)$ and $g \in L^r(0,1)$, then

$$g = \frac{\widehat{g}(0)}{2} e_0 + \sum_{j=1}^{\infty} \widehat{g}(j) e_j$$

where

$$\widehat{g}(k) := 2\int_0^1 g(x) e_k(x) \mathrm{d}x, \qquad \forall k \in \mathbb{N} \cup \{0\}$$

are the corresponding cosine Fourier coefficients. Moreover,

$$M_n g = \frac{\widehat{g}(0)}{2} e_0 + \sum_{j=1}^{\infty} \widehat{g}(j) M_n e_j = \frac{\widehat{g}(0)}{2} e_0 + \sum_{j=1}^{\infty} \widehat{g}(j) e_{nj} \in L^r(0,1).$$

Now, let $f_n(x) = \cos_p(n\pi_p x)$. Note that $e_0(x) = f_0(x) = 1$ for all $x \in \mathbb{R}$. Suitable linear extensions of the map $T : e_n \mapsto f_n$ are the changes of coordinates between $\{e_n\}_{n=0}^{\infty}$ and $\{f_n\}_{n=0}^{\infty}$. Our next goal is to find a canonical decomposition for T in terms of M_n and the Fourier coefficients $b_n(p)$. After that, we show that these are bounded operators of the Banach spaces $L^r(0,1)$ for all $r \in (1,\infty)$.

We have $\widehat{f_1}(k) = b_k(p)$ for all $k \in \mathbb{N} \cup \{0\}$. Since all of the functions $f_n(x)$ are continuous, then they all have a Fourier cosine expansion

$$f_n(x) = \frac{1}{2}\widehat{f_n}(0) e_0(x) + \sum_{k=1}^{\infty} \widehat{f_n}(k) e_k(x)$$

which is both pointwise convergent for all $x \in [0,1]$ and also convergent in the norm of $L^r(0,1)$ for all $r \in (1,\infty)$ (Theorem 2.1 and Lemma 2.1 respectively). For all

$n > 1$,

$$\widehat{f_n}(k) = 2\int_0^1 f_1(nx)\cos(k\pi x)\mathrm{d}x$$
$$= \begin{cases} b_m(p) & \text{for } mn = k,\ m \equiv_2 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$T = \sum_{j=1}^{\infty} b_j(p) M_j. \tag{4.10}$$

By virtue of Proposition 4.1, statement (4.9) and the triangle inequality, it follows that the expression (4.10) is convergent in the operator norm of $L^r(0,1)$ and that $T : L^r(0,1) \longrightarrow L^r(0,1)$ is a bounded linear operator (see the proof of Lemma 3.2). Moreover,

$$Te_0 = \sum_{j=1}^{\infty} b_j M_j e_0 = \sum_{j=1}^{\infty} b_j e_0 = \sum_{j=1}^{\infty} b_j e_j(0) = \cos_p(\pi_p 0) = 1 = f_0$$

and

$$Te_n = \sum_{j=1}^{\infty} b_j M_j e_n = \sum_{j=1}^{\infty} \widehat{f_1}(j) e_{nj} = \sum_{k=1}^{\infty} \widehat{f_n}(k) e_k = f_n, \qquad \forall n \in \mathbb{N}.$$

These are the change of basis maps between $\mathfrak{E}_{\cos}$ and $\mathfrak{E}_{\cos_p}$.

Below we employ the *one-term* criterion, stated in Theorem 3.1, in order to determine the basis thresholds for the family $\mathfrak{E}_{\cos_p}$ claimed in Theorem 4.1.

$$\sum_{\substack{j=3 \\ j\equiv_2 1}}^{\infty} |b_j(p)| < |b_1(p)| \quad \Rightarrow \quad \begin{cases} \mathfrak{E}_{\cos_p} \text{ is a Schauder basis} \\ \text{of } L^r(0,1) \text{ for all } r \in (1,\infty). \end{cases} \tag{4.11}$$

4.4 Bases properties of $\mathfrak{E}_{\cos_p}$

Our result is that,

Theorem 4.1. *There exist $p_0 < \frac{3}{2}$ and $p_1 > \frac{11}{5}$, such that $\mathfrak{E}_{\cos_p}$ is a Schauder basis* *of $L^r(0,1)$ for all $r \in (1,\infty)$ and $p \in [p_0, p_1]$.*

The theorem improves previously known results. See Table 1.1 for more details. The proof is separated into two cases.

4.4.1 The case $p \in (1,2)$

Recall the expression for $\mathcal{M}_p$ given in (4.2) and consider the identity

$$\pi_p^2 \mathcal{M}_p = \frac{\pi^3}{\pi^2 - 8}. \tag{4.12}$$

Lemma 4.8. *There exists $p_0 \in (1,2)$ such that (4.12) holds true for $p = p_0$. Moreover,*

$$\pi_p^2 \mathcal{M}_p < \frac{\pi^3}{\pi^2 - 8}, \qquad \forall p \in (p_0, 2).$$

Proof. It will be enough to prove that $\pi_p^2 \mathcal{M}_p$ is a convex function of the parameter p for all $p \in (1,2)$. Indeed, since

$$\lim_{p\to 1^+} \pi_p^2 \mathcal{M}_p = \infty \qquad \text{and} \qquad \lim_{p\to 2^-} \pi_p^2 \mathcal{M}_p = \pi^2 < \frac{\pi^3}{\pi^2 - 8},$$

both statements will immediately follow from this property.

Firstly note that

$$\frac{\mathrm{d}}{\mathrm{d}p} \ln(p-1)^{\frac{p-1}{p}} = \frac{1}{p^2} \ln(p-1) + \frac{1}{p}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}p^2} \ln(p-1)^{\frac{p-1}{p}} = \frac{2-p}{p^2(p-1)} - 2\frac{\ln(p-1)}{p^3} > 0.$$

Then $\ln(p-1)^{\frac{p-1}{p}}$ is convex for $p \in (1,2)$.

Similarly, we have

$$\frac{\mathrm{d}}{\mathrm{d}p} \ln(2-p)^{\frac{2-p}{p}} = \frac{-2}{p^2} \ln(2-p) - \frac{1}{p}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}p^2}\ln(2-p)^{\frac{2-p}{p}} = \frac{4-p}{p^2(2-p)} + 4\frac{\ln(2-p)}{p^3} > 0.$$

Then, also $\ln(2-p)^{\frac{2-p}{p}}$ is convex for $p \in (1,2)$.

Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}p}[\ln \pi_p] = \frac{\pi\cot(\frac{\pi}{p})}{p^2} - \frac{1}{p}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}p^2}\ln \pi_p = \frac{(p^2+\pi^2)}{p^4} - \frac{2\pi}{p^3}\cot\left(\frac{\pi}{p}\right) + \frac{\pi^2}{p^4}\cot^2\left(\frac{\pi}{p}\right) > 0.$$

The latter is a consequence of the fact that $\cos\frac{\pi}{p} < 0$ and $\sin\frac{\pi}{p} > 0$. Hence, also $\ln \pi_p^2$ is convex for $p \in (1,2)$.

The convexity of the logarithm of each one of the multiplying terms in the expression for $\pi_p^2\mathcal{M}_p$, implies that $\ln(\pi_p^2\mathcal{M}_p)$ is convex for $p \in (1,2)$. This ensures that indeed $\pi_p^2\mathcal{M}_p$ is convex in the same segment and the validity of the statement is ensured. □

Corollary 4.1. *Let $p_0 \in (1,2)$ be such that* (4.12) *holds true for $p = p_0$. The family $\mathfrak{E}_{\cos_p}$ is a Schauder basis of $L^r(0,1)$ for all $r \in (1,\infty)$ and $p \in [p_0, 2]$.*

Proof. According to Lemma 4.6,

$$\sum_{\substack{j=3\\ j\equiv_2 1}}^{\infty} |b_j(p)| < \frac{8\pi_p\mathcal{M}_p}{\pi^2}\sum_{\substack{j=3\\ j\equiv_2 1}}^{\infty}\frac{1}{j^2} = \frac{\pi_p^2\mathcal{M}_p(\pi^2-8)}{\pi^2\pi_p}. \tag{4.13}$$

On the other hand, in view of Lemma 4.5 and Lemma 1.7(c), we have

$$\begin{aligned} b_1(p) = \frac{\pi}{\pi_p}a_1 &= \frac{4\pi}{\pi_p}\int_0^{\frac{1}{2}} \sin_p(\pi_p x)\sin(\pi x)\mathrm{d}x \\ &\geq \frac{4\pi}{\pi_p}\int_0^{\frac{1}{2}} \sin^2(\pi x)\mathrm{d}x = \frac{\pi}{\pi_p}. \end{aligned}$$

Then, Lemma 4.8 yields

$$\sum_{\substack{j=3\\ j\equiv_2 1}}^{\infty} |b_j(p)| < b_1(p)$$

for all $p \in [p_0, 2)$. By virtue of (4.11) the claimed conclusion follows. $\square$

Since

$$\pi_{\frac{4}{3}}^2 \mathcal{M}_{\frac{4}{3}} = \frac{\pi^2 3^{\frac{5}{4}} \sqrt{2}}{2} > \frac{\pi^3}{\pi^2 - 8}$$

and

$$\pi_{\frac{3}{2}}^2 \mathcal{M}_{\frac{3}{2}} = \frac{64\pi^2}{27\sqrt[3]{4}} < \frac{\pi^3}{\pi^2 - 8},$$

then $p_0 \in (\frac{4}{3}, \frac{3}{2})$. This settles the proof of Theorem 4.1 for $p \in (1, 2)$.

Remark 4.1. *An implementation of the Matlab function (fzero) with the default tolerance gives* $p_0 \approx 1.458801$ *as an approximated solution of* (4.12) *with all digits correct.*

Remark 4.2. *The threshold* p_0 *can be improved by simply applying Corollary 3.1 to the case* $\widehat{f}_j = b_j$ *using the estimate presented in Lemma 4.6,*

$$\pi_p \mathcal{M}_p - 2b_1(p) < \sum_{\substack{j=1 \\ j \equiv_2 1}}^{k} \left(\frac{8\pi_p \mathcal{M}_p}{\pi^2 j^2} - |b_j(p)| \right). \tag{4.14}$$

This results in achieving a larger segment $(\hat{p}_{0,1}, p_0)$ *for* p *such that the family* $\mathfrak{E}_{\cos_p}$ *is a Schauder basis of* $L^r(0,1)$, $r \in (1, \infty)$ *as illustrated in Figure 5.2. Numerically,* $\hat{p}_{0,1} \approx 1.2978$ *for* $k = 201$*.*

4.4.2 Case $p \in (2, \infty)$

Recall the following identities involving the Riemann Zeta function [18, 3.411, 9.522 & 9.524],

$$\zeta(q) = \frac{1}{\Gamma(q)} \int_0^\infty \frac{t^{q-1}}{e^t - 1} \mathrm{d}t \qquad \mathrm{Re}(q) > 1, \tag{4.15}$$

$$\sum_{\substack{j=1 \\ j \not\equiv_2 0}}^{\infty} \frac{1}{j^q} = \left(1 - \frac{1}{2^q}\right) \zeta(q) \tag{4.16}$$

and

$$\frac{\zeta'(q)}{\zeta(q)} = -\sum_{k=1}^{\infty} \frac{\Delta(k)}{k^q} \tag{4.17}$$

where

$$\Delta(k) = \begin{cases} \ln(r) & \text{if } k = r^m \text{ for some } r \text{ prime and } m \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.9. *Let*

$$t_0 = \frac{2(e^2 - 3e + 1)}{(e^2 - 2e - 1)}.$$

Then

$$\zeta\left(\frac{3}{2}\right) < \frac{2}{\sqrt{\pi}} \left(2\sqrt{2} \arctan \frac{1}{\sqrt{2}} + \frac{\pi^2}{6} + \frac{t_0^2}{4} - \frac{(t_0 - 1)^2}{2(e-1)^2} - \frac{t_0(e-2)+1}{e-1}\right). \tag{4.18}$$

Proof. Since $\Gamma(1 + \frac{1}{2}) = \frac{\sqrt{\pi}}{2}$, the representation (4.15) gives

$$\begin{aligned} \zeta\left(\frac{3}{2}\right) &= \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{t^{1/2}}{e^t - 1} \mathrm{d}t \\ &= \frac{2}{\sqrt{\pi}} \left(\int_0^1 + \int_1^\infty \frac{t^{1/2}}{e^t - 1} \mathrm{d}t \right) = \frac{2}{\sqrt{\pi}} (\mathrm{J}_1 + \mathrm{J}_2). \end{aligned}$$

We estimate separately upper bounds for J_1 and J_2.

The change of variables $t = u^2$, yields

$$\begin{aligned} \mathrm{J}_1 &= \int_0^1 \frac{t^{1/2}}{e^t - 1} \mathrm{d}t < \int_0^1 \frac{t^{1/2}}{t + \frac{t^2}{2}} \mathrm{d}t \\ &= \int_0^1 \frac{2u^2}{u^2 + \frac{u^4}{2}} \mathrm{d}u = 2\sqrt{2} \arctan \frac{1}{\sqrt{2}}. \end{aligned}$$

On the other hand, we know that $\zeta(2) = \int_0^\infty \frac{t}{e^t - 1} \mathrm{d}t = \frac{\pi^2}{6}$, so

$$\mathrm{J}_2 \le \int_1^\infty \frac{t}{e^t - 1} \mathrm{d}t = \frac{\pi^2}{6} - \int_0^1 \frac{t}{e^t - 1} \mathrm{d}t.$$

We find lower bound for the integral on the right hand side, by interpolating the curve $c(t) = \frac{t}{e^t - 1}$ at two points, $t = 0$ and $t = 1$. Firstly observe that $c(t) \to 1$ as

$t \to 0$, $c(t)$ is decreasing and $c''(t) \geq 0$ for $t \in [0,1]$. Let t_0 be as in the hypothesis and let

$$\tilde{c}(t) = \begin{cases} 1 - \frac{1}{2}t & 0 \leq t \leq t_0 \\ \frac{1}{(e-1)^2}(1-t) + \frac{1}{e-1} & t_0 \leq t \leq 1 \end{cases}$$

be the piecewise linear interpolant of $c(t)$ in the two segments $[0, t_0]$ and $[t_0, 1]$, which is continuous at t_0. Note that $\tilde{c}(t)$ and $c(t)$ are tangent at $t = 0$ and $t = 1$. Then

$$c(t) \geq \tilde{c}(t) \qquad \forall t \in [0,1].$$

Hence

$$\begin{aligned}\int_0^1 c(t)\mathrm{d}t &\geq \int_0^{t_0} \left(1 - \frac{1}{2}t\right) \mathrm{d}t + \int_{t_0}^1 \left(\frac{1}{(e-1)^2}(1-t) + \frac{1}{e-1}\right) \mathrm{d}t \\ &= -\frac{t_0^2}{4} + \frac{(t_0-1)^2}{2(e-1)^2} + \frac{t_0(e-2)+1}{e-1}.\end{aligned}$$

Thus

$$\mathrm{J}_2 \leq \frac{\pi^2}{6} + \frac{t_0^2}{4} - \frac{(t_0-1)^2}{2(e-1)^2} - \frac{t_0(e-2)+1}{e-1}.$$

Alongside with the upper bound above for J_1, this ensures the validity of the claimed statement. □

Now, consider the equation

$$\frac{2\pi_{p'}}{\pi^2(p-1)} \left[2 + \frac{\pi^2}{2}(p-2)\right] \left[\left(1 - \frac{1}{2^{p'}}\right)\zeta(p') - 1\right] = \frac{8}{\pi\pi_p}. \tag{4.19}$$

Lemma 4.10. *There exists $p_1 \in (\frac{11}{5}, 3)$ such that (4.19) holds true for $p = p_1$. Moreover,*

$$\frac{2\pi_{p'}}{\pi^2(p-1)} \left[2 + \frac{\pi^2}{2}(p-2)\right] \left[\left(1 - \frac{1}{2^{p'}}\right)\zeta(p') - 1\right] < \frac{8}{\pi\pi_p}, \qquad \forall p \in [2, p_1).$$

Proof. From Lemma 1.2 it follows that the identity (4.19) reduces to

$$\frac{\pi}{p^2 \sin^2(\frac{\pi}{p})} \left[2 + \frac{\pi^2}{2}(p-2)\right] \left[\left(1 - \frac{1}{2^{p'}}\right)\zeta(p') - 1\right] = 1. \tag{4.20}$$

Denote by $h(p)$ the left hand side of (4.20). Then $h:(1,\infty)\longrightarrow\mathbb{R}$ is continuous and

$$h(2)=\frac{\pi}{2}\left(\frac{\pi^2}{8}-1\right)<1.$$

Since

$$\zeta\left(\frac{3}{2}\right)>1+\frac{\sqrt{2}}{4}+\sqrt{3}\sum_{k=3}^{\infty}\frac{1}{k^2}=\frac{4+\sqrt{2}}{4}+\sqrt{3}\left(\frac{\pi^2}{6}-\frac{5}{4}\right),$$

we get

$$\begin{aligned}
h(3)&=\frac{\pi}{9\sin^2(\frac{\pi}{3})}\left[2+\frac{\pi^2}{2}\right]\left[\left(1-\frac{1}{2^{\frac{3}{2}}}\right)\zeta\left(\frac{3}{2}\right)-1\right]\\
&>\frac{\pi}{9\sin^2(\frac{\pi}{3})}\left[2+\frac{\pi^2}{2}\right]\left[\left(1-\frac{1}{2^{\frac{3}{2}}}\right)\left(\frac{4+\sqrt{2}}{4}+\sqrt{3}\left(\frac{\pi^2}{6}-\frac{5}{4}\right)\right)-1\right]\\
&>1.
\end{aligned}$$

Hence, there exists $p_1\in(2,3)$ such that $h(p_1)=1$.

The derivative

$$\frac{\mathrm{d}}{\mathrm{d}q}\left[\left(1-\frac{1}{2^q}\right)\zeta(q)\right]=\frac{\ln(2)}{2^q}\zeta(q)+\left(1-\frac{1}{2^q}\right)\zeta'(q)$$

is negative for any $q\in(1,2)$. Indeed the identity (4.17) gives

$$\begin{aligned}
\frac{\zeta'(q)}{\zeta(q)}&<-\frac{\ln(2)}{2^q}-\frac{\ln(3)}{3^q}-\frac{\ln(2)}{4^q}\\
&<-\ln(2)\left[\frac{1}{2^q}+\frac{1}{3^q}+\frac{1}{4^q}\right]<\frac{\ln(2)}{1-2^q},
\end{aligned}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}q}\left[\left(1-\frac{1}{2^q}\right)\zeta(q)\right]=\zeta(q)\left[\frac{\ln(2)}{2^q}+\frac{2^q-1}{2^q}\frac{\zeta'(q)}{\zeta(q)}\right]<0.$$

Since p' and $\sin\left(\frac{\pi}{p}\right)$ are decreasing functions of $p\in(2,\infty)$, then

$$\frac{\pi}{\sin^2(\frac{\pi}{p})}\left[\left(1-\frac{1}{2^{p'}}\right)\zeta(p')-1\right]$$

is an increasing function of $p\in(2,\infty)$.

As

$$\frac{\mathrm{d}}{\mathrm{d}p}\left[\frac{1}{p^2}\Big(2+\frac{\pi^2}{2}(p-2)\Big)\right] = \frac{1}{p^3}(-\frac{\pi^2}{2}p+2\pi^2-4) > 0, \qquad \forall p \in [2,3]$$

then $h(p)$ is increasing for $p \in [2,3]$ and so indeed

$$h(p) < h(p_1) = 1, \qquad \forall p \in [2, p_1).$$

Let us now show that $p_1 > \frac{11}{5}$. Let c_1 denote the right hand side of the estimate (4.18) in Lemma 4.9. Since $\zeta(q)$ is convex in the segment $[\frac{3}{2}, 2]$, that is, the straight line joining the points $(\frac{3}{2}, c_1)$ and $(2, \frac{\pi^2}{6})$ is above the curve $\zeta(q)$ for all $q \in [\frac{3}{2}, 2]$. Then,

$$\zeta(q) \le \left(\frac{\pi^2}{3} - 2c_1\right)(q-2) + \frac{\pi^2}{6}$$

and

$$\zeta\left(\frac{11}{6}\right) \le \frac{\pi^2}{9} + \frac{c_1}{3}. \tag{4.21}$$

Note that for $p = \frac{11}{5}$, $p' = \frac{11}{6}$. Now, $\sin(\pi y)$ is concave for $y \in [\frac{5}{12}, \frac{1}{2}]$. Then it is above the straight line joining the points $(\frac{5}{12}, \sin\frac{5\pi}{12})$ and $(\frac{1}{2}, 1)$. That is

$$\sin(\pi y) \ge \left(12 - 12\sin\frac{5\pi}{12}\right)\left(y - \frac{1}{2}\right) + 1, \qquad \forall y \in \left[\frac{5}{12}, \frac{1}{2}\right].$$

Then

$$\sin\frac{5\pi}{11} > \frac{\sqrt{6}}{22}\left(\sqrt{3}+3\right) + \frac{5}{11}. \tag{4.22}$$

Denote by c_2 the right hand side of the latter inequality. From (4.21) and (4.22), it follows that

$$\begin{aligned} h\left(\frac{11}{5}\right) &= \frac{\pi}{(\frac{11}{5})^2\sin^2(\frac{5\pi}{11})}\left[2+\frac{\pi^2}{2}\left(\frac{11}{5}-2\right)\right]\left[\left(1-\frac{1}{2^{11/6}}\right)\zeta\left(\frac{11}{6}\right)-1\right] \\ &< \frac{\pi}{\frac{121}{25}c_2^2}\left(2+\frac{\pi^2}{10}\right)\left[\left(1-\frac{1}{2^{11/6}}\right)\left(\frac{\pi^2}{9}+\frac{c_1}{3}\right)-1\right] < 1. \end{aligned}$$

As $h(p)$ is increasing, then indeed $p_1 > \frac{11}{5}$. □

Corollary 4.2. *Let $p_1 \in (2, \infty)$ be such that (4.19) holds true for $p = p_1$. The* *family $\mathfrak{E}_{\cos_p}$ forms a Schauder basis of $L^r(0,1)$ for all $r \in (1,\infty)$ and $p \in [2, p_1]$.*

Proof. From Lemma 4.7 and (4.16), we have

$$\sum_{\substack{j=3 \\ j\equiv_2 1}}^{\infty} |b_j(p)| < \frac{2\pi_{p'}}{\pi^2(p-1)}\left[2+\frac{\pi^2}{2}(p-2)\right]\left[\left(1-\frac{1}{2^{p'}}\right)\zeta(p')-1\right]. \tag{4.23}$$

According to part "b" of Lemma 4.1, $\sin_p(\pi_p x)$ is strictly concave on $(0,\frac{1}{2})$. Then

$$\begin{aligned} a_1(p) &= 2\int_0^1 \sin_p(\pi_p x)\sin(\pi x)\mathrm{d}x = 4\int_0^{\frac{1}{2}} \sin_p(\pi_p x)\sin(\pi x)\mathrm{d}x \\ &> 4\int_0^{\frac{1}{2}} (2x)\sin(\pi x)\mathrm{d}x = \frac{8}{\pi^2}. \end{aligned}$$

Hence, in view of Lemma 4.5, we get

$$b_1(p) = \frac{\pi}{\pi_p}a_1(p) > \frac{8}{\pi\pi_p}. \tag{4.24}$$

From Lemma 4.10, it then follows that

$$\sum_{\substack{j=3 \\ j\equiv_2 1}}^{\infty} |b_j(p)| < b_1(p), \qquad \forall p \in [2,p_1].$$

By virtue of (4.11) this implies the claimed conclusion. □

Remark 4.3. *An approximation of the solution of* (4.19) *using the Matlab function (fzero) with the default tolerance gives* $p_1 \approx 2.42865$ *with all digits correct.*

Remark 4.4. *The threshold p_1 can be improved by simply applying Corollary 3.1 to the case $\widehat{f_j} = b_j$ using the estimate presented in Lemma 4.7,*

$$\frac{2\pi_{p'}}{\pi^2(p-1)}\left[2+\frac{\pi^2}{2}(p-2)\right]\left(\sum_{\substack{j=1 \\ j\equiv_2 1}}^{k} j^{-p'} - (1-2^{-p'})\zeta(p')\right) > \sum_{j=1}^{k}|b_j(p)| - 2b_1(p). \tag{4.25}$$

Consequently, we arrive at a larger segment $(p_1, \hat{p}_{1,1})$ for p such that the family $\mathfrak{E}_{\cos_p}$ is a Schauder basis of $L^r(0,1)$, $r \in (1,\infty)$ as illustrated in Figure 5.3. Numerically,

$\hat{p}_{1,1} \approx 3.2205$ *for* $k = 201$.

4.5 Bases properties of $\mathfrak{E}_{\sin_p}$

The best interval for Schauder basis for the family $\mathfrak{E}_{\sin_p}$ was given in [7] to be $[\tilde{p}_1, \infty)$ where $\tilde{p}_1 \approx 1.087$ (Section 2.1.1), using the estimate:

$$|a_j(p)| < \phi_j = \frac{4\pi_p}{\pi^2 j^2}, \qquad \forall j \geq 1. \tag{4.26}$$

Here we apply Corollary 3.1, the improved version of the so called *one-term* criterion, in the context of Section 3.1.

Corollary 4.3. *If*

$$\frac{\pi_p}{2} - 2a_1(p) < \sum_{\substack{j=1 \\ j \equiv_2 1}}^{k} \left(\frac{4\pi_p}{\pi^2 j^2} - |a_j(p)| \right), \tag{4.27}$$

then $\mathfrak{E}_{\sin_p}$ *is a Schauder basis of* $L^r(0,1)$ *for* $r \in (1, \infty)$.

Proof. In view of (4.26). The proof is an immediate consequence of Corollary 3.1 when it is expressed in terms of the Fourier coefficients $a_j(p)$ of $\sin_p$. □

This Corollary improves the threshold of invertibility of the operator T which guarantees the set $\mathfrak{E}_{\sin_p}$ forming a Schauder basis of $L^r(0,1)$ for $r \in (1, \infty)$ and $p \in (\tilde{p}_{1,1}, \infty)$ with $\tilde{p}_{1,1} \approx 1.0484$ for $k = 141$ (see, Figure 5.1 and Remark 5.2 below).

4.6 The regularity of the p-trigonometric functions

Definition 4.1. *Let* $s \geq 0$ *be a real number. Denote by* $\mathcal{L}^{s,r}(\mathbb{R})(1 < r < \infty)$ *the class of all functions* F *such that* $F = G_s * f$ *for* $f \in L^r(\mathbb{R})$, *where* G_s *is the Bessel kernel of order* s *with Fourier transform* $\widehat{G}_s(\xi) = (1 + |\xi|^2)^{-s/2}$, $\xi \in \mathbb{R}$. *The space is called the space of Bessel potentials. In general it is a Banach space with norm*

$\|F\|_{s,r} := \|f\|_{L^r(\mathbb{R})}$ *and a Hilbert space in the special case* $r = 2$, $H^s(\mathbb{R}) \equiv \mathcal{L}^{s,2}(\mathbb{R})$ *equipped with the norm*

$$\|F\|_{s,2}^2 = \sum_{j=-\infty}^{\infty} (1 + j^2)^s |\widehat{F}(j)|^2,$$

where $\widehat{F}(j)$ *is the* j*-th Fourier coefficient of* F*. For* $s = k \in \mathbb{N} \cup \{0\}$*, the space is identical to the Sobolev space* $\mathcal{L}^{k,r}(\mathbb{R}) \equiv W^{k,r}(\mathbb{R})$ *and the corresponding norms are equivalent.*

In what follows, if $(c_k)_k$ and $(d_k)_k$ are non-negative sequences of real numbers, we use the notation $c_k \lesssim d_k$ to indicate the existence of a constant $C > 0$ independent of k such that $c_k \leq C d_k$ for all $k \in \mathbb{N}$.

Let $p \in (1, 2)$. According to the formula [9, (4.4)], it follows that the Fourier coefficients of the p-sine function are such that

$$|a_j(p)| \leq \frac{16\pi_p^2 \mathcal{M}_p}{\pi^3} j^{-3}, \qquad \forall j \in \mathbb{N}.$$

Then, $\sin_p(\pi_p \cdot) \in H^s(0, 1)$ for all $s < \frac{5}{2}$.

Numerical estimates for the Sobolev regularity of $\sin_p(\pi_p \cdot)$ for $2 < p < 100$ were reported in [6, Figure 2]. From that picture, one may conjecture that for $p > 3$, $\sin_p(\pi_p \cdot) \notin H^2(0, 1)$. Moreover, the regularity appears to drop asymptotically to $\frac{3}{2}$ for p large. By contrast, it appears that $\sin_p(\pi_p \cdot) \in H^2(0, 1)$ for $p \in (2, 3)$. The following statement, which is a consequence of Lemma 4.7, settles this conjecture.

Corollary 4.4. *For* $p \in (2, \infty)$ *set* $r(p) = p' + \frac{1}{2}$*. Then* $\sin_p(\pi_p \cdot) \in H^s(0, 1)$ *for all* $s \in [0, r(p))$*.*

Proof. According to Lemma 4.5,

$$|a_j(p)| = \frac{\pi_p}{j\pi} |b_j(p)|.$$

Then, by virtue of Lemma 4.7,

$$|a_j(p)| \leq \frac{2\pi_p \pi_{p'}}{\pi^3 (p-1)} \left[2 + \frac{\pi^2}{2}(p-2) \right] j^{-(p'+1)}, \qquad \forall j \geq 3.$$

Let $\langle j\rangle^2 = 1 + j^2$. For $s < p' + \frac{1}{2}$,

$$\sum_{j=1}^{\infty} \langle j\rangle^{2s} |a_j(p)|^2 \lesssim 2^s a_1^2(p) + \sum_{\substack{j=3 \\ j\equiv_2 1}}^{\infty} \frac{1}{j^{1+\epsilon(p)}} < \infty$$

where

$$\epsilon(p) = 1 - 2s + 2p' > 0.$$

Hence $\sin_p(\pi_p \cdot) \in H^s(0,1)$ as claimed. $\square$

In this final section we describe various connections between the statements established above and those reported in the literature.

4.7 Connection with other work

The recent papers [12] and [8] seem to be the only ones in the existing literature which conduct an analysis of the basis properties of the p-cosine functions. In the notation of [12] we fix $\alpha = 1$ and $p = q > 1$. The Fourier coefficients of the p-cosine functions are

$$\eta_j(p,p) = b_j(p), \qquad \forall j \in \mathbb{N} \cup \{0\}.$$

The condition [12, (2.2)] as well as the criterion for determining whether $\mathfrak{C}_{\cos_p}$ is a Schauder basis of $L^r(0,1)$ are exactly the same as (4.11). Let us compare some of the results of [12] with those of this thesis.

In [12, Proposition 2.5], the estimate [12, (2.20)] is equivalent to the following. There exists $p_0^* = \frac{72(\pi-2)-2\pi^3}{96(\pi-2)-3\pi^3}$, such that

$$\eta_1(p,p) \geq \begin{cases} \frac{\pi(p-1)}{2p-1} - \frac{(\pi-2)(p-1)}{3p-2} & p \in (1, p_0^*) \\ \frac{\pi(p-1)}{2p-1} - \frac{\pi^3(p-1)}{24(4p-3)} & p \in (p_0^*, \infty). \end{cases} \tag{4.28}$$

Here p_0^* satisfies the identity

$$\frac{4p-3}{3p-2} = \frac{\pi^3}{24(\pi-2)}.$$

Note that $p_0^* \approx 1.22$.

Let us consider firstly the regime $p \in (1,2)$. From [12, Proposition 2.2] it follows that

$$\sum_{k=1}^{\infty} |\eta_{2k+1}(p,p)| \leq \frac{\pi_p(\pi^2-8)}{\pi^2} \qquad \forall p \in (1,2). \tag{4.29}$$

As $\mathcal{M}_p < 1$ whenever $p \in (1,2)$ in (4.2), then (4.13) is sharper than (4.29) in this regime.

If $p \in (1, p_0^*)$, then

$$\frac{\pi_p(\pi^2-8)}{\pi^2} > \frac{\pi(p-1)}{2p-1} - \frac{(\pi-2)(p-1)}{3p-2},$$

and no conclusion about the validity of (4.11) can be derived in this case from (4.28) and (4.29). For $p \in (p_0^*, 2)$, on the other hand,

$$\frac{\pi_p(\pi^2-8)}{\pi^3} < \frac{p-1}{2p-1} - \frac{\pi^2(p-1)}{24(4p-3)} \qquad \Longleftrightarrow \qquad p \in (p_0^\dagger, 2),$$

where $p_0^\dagger \approx 1.75$. In order to see this, note that π_p is decreasing and $\lim_{p\to 1^+} \pi_p = \infty$, while the right hand side of this identity is increasing for $p \in (1,2)$. Thus, a combination of [12, Proposition 2.2] and [12, Proposition 2.5], only guarantees that $\mathfrak{E}_{\cos_p}$ is a Schauder basis of $L^r(0,1)$ for $p \in [p_0^\dagger, 2)$ where $p_0^\dagger > \frac{3}{2} > \hat{p}_{0,1}$.

As it turns, it is not possible to deduce from the results of [12] any basis property of the family $\mathfrak{E}_{\cos_p}$ in the complementary regime $p \in (2,\infty)$. Here is how the different estimates on the Fourier coefficients compare in this case.

From [12, Proposition 2.4], we gather that

$$\sum_{k=1}^{\infty} |\eta_{2k+1}(p,p)| \leq \frac{2\pi_{p'}}{\pi^2(p-1)} \left[4+\pi(p-1)\right] \left[\left(1-\frac{1}{2^{p'}}\right)\zeta(p')-1\right]. \tag{4.30}$$

Since

$$4+\pi(p-1) \geq 2+\frac{\pi^2}{2}(p-2) \qquad \forall p \leq \frac{4+2\pi^2-2\pi}{\pi^2-2\pi},$$

the upper bound (4.23) is sharper than (4.30) for $p \in [2,3]$. The latter is the relevant regime in the proof of Theorem 4.1.

Since $\pi_p < \pi$ for $p \in (2,\infty)$, the lower bound (4.24) is sharper than [12, (2.19)].

Moreover,

$$\frac{8}{\pi\pi_p} > \frac{\pi(p-1)}{2p-1} - \frac{\pi^3(p-1)}{24(4p-3)} \qquad \forall p \in (2,\infty).$$

Hence the estimate (4.28), which is [12, (2.20)], is also superseded by (4.24) for $p \in (2,\infty)$.

Chapter 5

Riesz Basis Properties of p-trigonometric Functions

This chapter shall examine the new *multi-term* criteria introduced in Chapter 3 when applied to p-trigonometric functions: p-sine, p-cosine and p-exponential. Our results improve upon those of Chapter 4, [12] and [7] for $r = 2$.

5.1 Fourier coefficients

For all $p \in (1, \infty)$, consider the definitions of $a_j(p)$ and $b_j(p)$ given by (4.3) and (4.4), respectively. Integration by parts, changing the variable of integration to $t = \sin_p(\pi_p x)$ and using (1.15) yield,

$$\begin{aligned} a_j(p) &= 4 \int_0^{\frac{1}{2}} \sin_p(\pi_p x) \sin(j\pi x) \mathrm{d}x \\ &= \frac{4}{j\pi} \int_0^{\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}x} \sin_p(\pi_p x) \cos(j\pi x) \mathrm{d}x \\ &= \frac{4}{j\pi} \int_0^1 \cos\left[\frac{j\pi}{2} I_p(t)\right] \mathrm{d}t. \end{aligned}$$

Hence

$$a_j(p) = \begin{cases} 0 & j \equiv_2 0 \\ \frac{4}{j\pi} \int_0^1 \cos\left[\frac{j\pi}{2} I_p(t)\right] \mathrm{d}t & j \equiv_2 1. \end{cases} \tag{5.1}$$

By virtue of Lemma 4.5,

$$b_j(p) = \begin{cases} 0 & j \equiv_2 0 \\ \frac{4}{\pi_p} \int_0^1 \cos\left[\frac{j\pi}{2} I_p(t)\right] \mathrm{d}t & j \equiv_2 1. \end{cases} \tag{5.2}$$

5.1.1 Estimates for $a_j(p)$

In order to determine improved thresholds for the known bases properties of the families $\mathfrak{E}_{\sin_p}$ and $\mathfrak{E}_{\cos_p}$, we establish several estimates for the Fourier coefficients of $\sin_p$ and $\cos_p$.

We begin by examining the Fourier coefficients $a_3(p)$ and $a_9(p)$ as p increases. Various other technical points are included in Appendices A.2 and A.3 with some Matlab codes in B.1.1 and B.1.2. The results obtained give numerical lower bounds for $a_3(p)$ and $a_9(p)$ as shown in Tables 5.1 and 5.2, respectively.

5.1.2 Estimates for $b_j(p)$

We now establish a comparison between $b_1(p)$, $b_3(p)$ and $b_9(p)$. This is analogous to [7, Lemma 4.2] for p-cosine.

Lemma 5.1. *Let $j = 3, 9$, we have $b_j(p) < b_1(p)$.*

Proof. For $j = 3$ and $p > 1$. Observe that $\cos_p(\pi_p x) > 0$ for all $x \in [0, 1/2)$ and $\cos(\pi x) - \cos(3\pi x) = 2\sin(2\pi x)\sin(\pi x) > 0$ whenever $x \in (0, 1/2)$. Then,

$$\begin{aligned} b_1(p) - b_3(p) &= 2\int_0^1 \cos_p(\pi_p x)(\cos(\pi x) - \cos(3\pi x))\mathrm{d}x \\ &= 4\int_0^{\frac{1}{2}} \cos_p(\pi_p x)(\cos(\pi x) - \cos(3\pi x))\mathrm{d}x > 0, \end{aligned}$$

and the result follows.

Now, let $j = 9$. The zeros of the trigonometric equation

$$\cos(\pi x) - \cos(9\pi x) = 2\sin(5\pi x)\sin(4\pi x) = 0, \qquad x \in [0, 1/2]$$

are $x_0 = 0$, $x_1 = \frac{1}{5}$, $x_2 = \frac{1}{4}$, $x_3 = \frac{2}{5}$ and $x_4 = \frac{1}{2}$.

λ	m_1^-	m_1^+	a_3 lower bound
1.5	2	3	0.0692320
1.5	3	3	0.0912921
1.5	4	3	0.0996541
1.9	3	3	0.00534857

Table 5.1: Lower bound estimates for $a_3(p)$ when $p \in (1, \lambda]$ (see Appendix B.1.1)

λ	m_1^-	m_1^+	m_2^-	m_2^+	m_3^-	a_9 lower bound
1.5	4	5	5	4	2	8.76881e-06
1.5	5	5	5	4	2	8.35771e-05

Table 5.2: Lower bound estimates for $a_9(p)$ when $p \in (1, \lambda]$ (see Appendix B.1.2)

For $k \in \{0, 1, 2, 3\}$. Set

$$\mathcal{I}_k = \int_{x_k}^{x_{k+1}} \cos_p(\pi_p x)(\cos(\pi x) - \cos(9\pi x))\mathrm{d}x.$$

Observe that

$$\cos(\pi x) > \cos(9\pi x), \qquad \forall x \in (x_0, x_1) \cup (x_2, x_3)$$

and

$$\cos(\pi x) < \cos(9\pi x), \qquad \forall x \in (x_1, x_2) \cup (x_3, x_4).$$

We conclude that $\mathcal{I}_k > 0$ for $k \in \{0, 2\}$ and $\mathcal{I}_k < 0$ for $k \in \{1, 3\}$. Moreover, the following inequality

$$|\cos(\pi x) - \cos(9\pi x)| < \cos(\pi(x - 1/9)) - \cos(9\pi(x - 1/9))$$

holds whenever $x \in (3/18, 5/18) \cup (7/18, 1/2)$. Hence,

$$
\begin{aligned}
|\mathcal{I}_1| &\leq \int_{x_1}^{x_2} \cos_p(\pi_p x)|\cos(\pi x) - \cos(9\pi x)|\mathrm{d}x \\
&< \int_{x_1}^{x_2} \cos_p(\pi_p x)\big(\cos(\pi(x-1/9)) - \cos(9\pi(x-1/9))\big)\mathrm{d}x
\end{aligned}
$$

and

$$
\begin{aligned}
|\mathcal{I}_3| &\leq \int_{x_3}^{x_4} \cos_p(\pi_p x)|\cos(\pi x) - \cos(9\pi x)|\mathrm{d}x \\
&< \int_{x_3}^{x_4} \cos_p(\pi_p x)\big(\cos(\pi(x-1/9)) - \cos(9\pi(x-1/9))\big)\mathrm{d}x.
\end{aligned}
$$

Change variable to $t = x - \frac{1}{9}$. Notice that for any $t \in (4/45, 5/36) \cup (13/45, 7/18)$,

$$
\cos_p(\pi_p(t+1/9)) < \cos_p(\pi_p t).
$$

Hence,

$$
\begin{aligned}
|\mathcal{I}_1| &< \int_{x_1-1/9}^{x_2-1/9} \cos_p(\pi_p(t+1/9))(\cos(\pi t) - \cos(9\pi t))\mathrm{d}t \\
&= \int_{\frac{4}{45}}^{\frac{5}{36}} \cos_p(\pi_p(t+1/9))(\cos(\pi t) - \cos(9\pi t))\mathrm{d}t \\
&< \int_{\frac{4}{45}}^{\frac{5}{36}} \cos_p(\pi_p t)(\cos(\pi t) - \cos(9\pi t))\mathrm{d}t \\
&< \int_{x_0}^{x_1} \cos_p(\pi_p t)(\cos(\pi t) - \cos(9\pi t))\mathrm{d}t = \mathcal{I}_0.
\end{aligned}
$$

The last two inequalities are due to the fact that $x_0 < 4/45$, $x_1 > 5/36$ and $\cos(\pi t) - \cos(9\pi t) > 0$ for $t \in (x_0, x_1)$. Also

$$
\begin{aligned}
|\mathcal{I}_3| &< \int_{x_3-1/9}^{x_4-1/9} \cos_p(\pi_p(t+1/9))(\cos(\pi t) - \cos(9\pi t))\mathrm{d}t \\
&= \int_{\frac{13}{45}}^{\frac{7}{18}} \cos_p(\pi_p(t+1/9))(\cos(\pi t) - \cos(9\pi t))\mathrm{d}t \\
&< \int_{\frac{13}{45}}^{\frac{7}{18}} \cos_p(\pi_p t)(\cos(\pi t) - \cos(9\pi t))\mathrm{d}t \\
&< \int_{x_2}^{x_3} \cos_p(\pi_p t)(\cos(\pi t) - \cos(9\pi t))\mathrm{d}t = \mathcal{I}_2,
\end{aligned}
$$

and this is because $x_2 < 13/45$, $x_3 > 7/18$ and $\cos(\pi t) - \cos(9\pi t) > 0$ for $t \in (x_2, x_3)$.

Consequently,

$$\begin{aligned}
-\mathcal{I}_1 &= |\mathcal{I}_1| < |\mathcal{I}_0| = \mathcal{I}_0 \\
-\mathcal{I}_3 &= |\mathcal{I}_3| < |\mathcal{I}_2| = \mathcal{I}_2.
\end{aligned}$$

Then,

$$\begin{aligned}
b_1(p) - b_9(p) &= 4 \int_0^{\frac{1}{2}} \cos_p(\pi_p x)(\cos(\pi x) - \cos(9\pi x))\mathrm{d}x \\
&= 4 \sum_{k=0}^{3} \mathcal{I}_k > 4(-\mathcal{I}_1 + \mathcal{I}_1 - \mathcal{I}_3 + \mathcal{I}_3) = 0,
\end{aligned}$$

and the result follows. □

5.2 Bases properties of $\mathfrak{E}_{\sin_p}$

The best threshold for Riesz basis (case $r = 2$) for the family $\mathfrak{E}_{\sin_p}$ was given in [7] to be at least $\hat{p}_1 \approx 1.04392$ using the estimate (4.26).

Here we employ the new *multi-term* criteria in order to examine the set of p-values for which the family $\mathfrak{E}_{\sin_p}$ is guaranteed to generate a basis of $L^2(0,1)$, and hence to investigate whether this approach will improve on the results of [7].

Corollary 5.1. *Let* $3 \leq k \equiv_2 1$ *and* $p > 1$. *Suppose that* $a_9(p) > 0$ *and* $|a_3(p)(a_9(p) + a_1(p))| \geq 4|a_9(p)a_1(p)|$. *If*

$$\frac{4\pi_p}{\pi^2}\left[\pi^2/8 - \sum_{\substack{j=1 \\ j\equiv_2 1}}^{k} (1/j^2)\right] < a_1(p) + 2a_9(p) - \sum_{j=3}^{k} |a_j(p)|, \tag{5.3}$$

then $\mathfrak{E}_{\sin_p}$ *is a Riesz basis of* $L^2(0,1)$.

Proof. For $f(x) = \sin_p(\pi_p x), d = 1$ and $\mathfrak{p} = 3$ we have $\mathcal{F} = \{1, 3, 9\}$. Then $m(z) = a_1(p) + a_3(p)z + a_9(p)z^2$ for $z \in \mathbb{D}$. According to Lemma 2.3, condition (3.23) holds. Then, since $|a_3(p)(a_9(p) + a_1(p))| \geq 4|a_9(p)a_1(p)|$ we have $\omega = a_1(p) + a_9(p) - |a_3(p)|$ (see the proof of Corollary 3.3(a)). In view of (4.26) and (5.3), the statement (3.22)

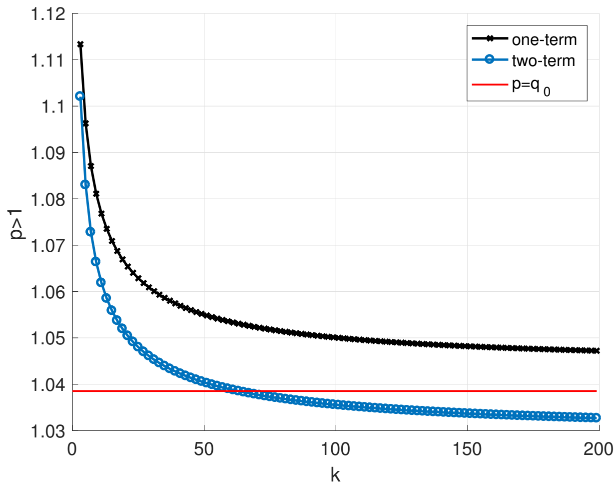

Figure 5.1: Two plots for the zeros of (4.27) and (5.3), considered as equations in p, against $k \in \mathbb{N}$. Using the improved *one-term* Schauder basis criterion and the *two-term* Riesz basis criterion, respectively, we determine thresholds for $\mathfrak{E}_{\sin_p}$ to be a basis when $p \in (1, \infty)$.

is fulfilled. Hence Corollary 3.2 ensures the conclusion. $\square$

Numerically we see that the inequality $|a_3(p)(a_9(p) + a_1(p))| \geq 4|a_9(p)a_1(p)|$ holds for any $p \in (q_0, 2)$ where $q_0 \approx 1.038537$ and $a_9(p) > 0$ for $p \in (1, 1.5]$. On the other hand (5.3) is fulfilled for any $p \in (\tilde{p}_0, 2)$ where $\tilde{p}_0 \approx 1.032661 < q_0$ given that $k = 201$.

Remark 5.1. *By combining Corollary 5.1 with the results of Section 7 in [7] (mentioned briefly in Section 2.1.1) it immediately follows that the threshold for invertibility of the operator T, defined by* (3.4)*, in the Hilbert space setting is at least $q_0 < \hat{p}_1$. Hence. the family $\mathfrak{E}_{\sin_p}$ forms a Riesz basis of $L^2(0,1)$ for all $p \in (q_0, \infty)$.*

Remark 5.2. *The graph on the left in Figure 5.1 shows improvements in the p-threshold for larger values of k. According to [9], the one-term criterion ceases to be valid for $p \approx 1.04399$. This implies that Corollary 4.3 does not go beyond this value. The graph of the two-term criterion (right) decays faster than the former. Here it provides improved thresholds for smaller values of k. This is an indication of the efficiency of the new multi-term criteria that we develop in this thesis.*

5.3 Bases properties of $\mathfrak{E}_{\cos_p}$

Consider upper bound estimates of the Fourier coefficients $b_j(p)$. See Lemmas 4.6, 4.7 and statement (5.2),

$$|b_j(p)| < \phi_j$$

such that

$$\phi_j = \begin{cases} \frac{8\pi_p}{j^2\pi^2}\mathcal{M}_p & p \in (1,2),\ j \geq 1 \\ \frac{4}{\pi_p} & p \in (2,\infty),\ j = 1 \\ \frac{2\pi_{p'}}{\pi^2(p-1)}\left[2 + \frac{\pi^2}{2}(p-2)\right] j^{-p'} & p \in (2,\infty),\ j \geq 3 \end{cases} \tag{5.4}$$

where $\mathcal{M}_p := (p-1)^{\frac{p-1}{p}}(2-p)^{\frac{2-p}{p}}$.

5.3.1 Case $p \in (1,2)$

From (4.12), we know that the equation

$$\pi_p^2\mathcal{M}_p = \frac{\pi^3}{\pi^2 - 8}$$

has a root at $p = p_0 \approx 1.458801$ (see Remark 4.1). This is the best p-threshold achieved by Theorem 3.1 (for even functions) in the case $p \in (1,2)$ such that $\mathfrak{E}_{\cos_p}$ is a basis in $L^r(0,1)$ for all $p \in [p_0, 2]$.

Lemma 5.2. *There exists* $p_{0,1} \in (1, p_0)$ *such that*

$$\pi_p^2\mathcal{M}_p = \frac{\pi^3}{\pi^2 - 8 - 8/81}. \tag{5.5}$$

Moreover,

$$\pi_p^2\mathcal{M}_p \leq \frac{\pi^3}{\pi^2 - 8 - 8/81}, \qquad \forall p \in [p_{0,1}, p_0]. \tag{5.6}$$

Proof. Since

$$\frac{\pi^3}{\pi^2 - 8} < \frac{\pi^3}{\pi^2 - 8 - 8/81},$$

the proof follows exactly in the same way as the proof of Lemma 4.8.

□

Corollary 5.2. *Let* $p_{0,1} \in (1,2)$ *be such that* (5.5) *holds,* $b_9(p) > 0$ *and* $|b_3(p)(b_9(p)+b_1(p))| \geq 4|b_9(p)b_1(p)|$. *Then for any* $p \in [p_{0,1}, p_0]$ *the family* $\mathfrak{E}_{\cos_p}$ *is a Riesz basis of* $L^2(0,1)$.

Proof. According to Lemma 5.1, statement (3.23) holds with $\mathfrak{p} = 3$ and $f(x) = \cos_p(\pi_p x)$. By virtue of (5.4) we have

$$\sum_{j\in\mathbb{N}\setminus\{1,9\}}^{\infty} |b_j(p)| < \frac{8\pi_p\mathcal{M}_p}{\pi^2} \sum_{\substack{j\in\mathbb{N}\setminus\{1,9\} \\ j\equiv_2 1}}^{\infty} \frac{1}{j^2} = \frac{\pi_p\mathcal{M}_p}{\pi^2}(\pi^2 - 8 - 8/81).$$

Since $\sin_p$ is non-increasing in p for fixed x (Lemma 1.7(c)), then

$$\begin{aligned} a_1(p) &= 4\int_0^{\frac{1}{2}} \sin_p(\pi_p x)\sin(\pi x)\mathrm{d}x \\ &> 4\int_0^{\frac{1}{2}} \sin^2(\pi x)\mathrm{d}x = 1. \end{aligned}$$

Thus, in view of Lemma 4.5 we get

$$b_1(p) + b_9(p) > b_1(p) = \frac{\pi}{\pi_p}a_1(p) > \frac{\pi}{\pi_p}.$$

Hence because of (5.6),

$$\sum_{j\in\mathbb{N}\setminus\{1,9\}}^{\infty} |b_j(p)| < b_1(p) + b_9(p).$$

By virtue of Corollary 3.3*(a)* the assertion follows. □

Lemma 5.2 together with Corollary 5.2 provide an analytical tool for improving Corollary 4.1. Numerically we can show that $|b_3(p)(b_9(p)+b_1(p))| \geq 4|b_9(p)b_1(p)|$ holds for all $p \in (q_1, 2)$ where $q_1 \approx 1.128298$ is the root of the equation $|b_3(p)(b_9(p)+b_1(p))| = 4|b_9(p)b_1(p)|$. Also $b_9(p) > 0$ for all $p \in (1, 1.5]$. Moreover the solution $p_{0,1}$ of (5.5) is approximately 1.441908 which belongs to the interval (q_1, p_0).

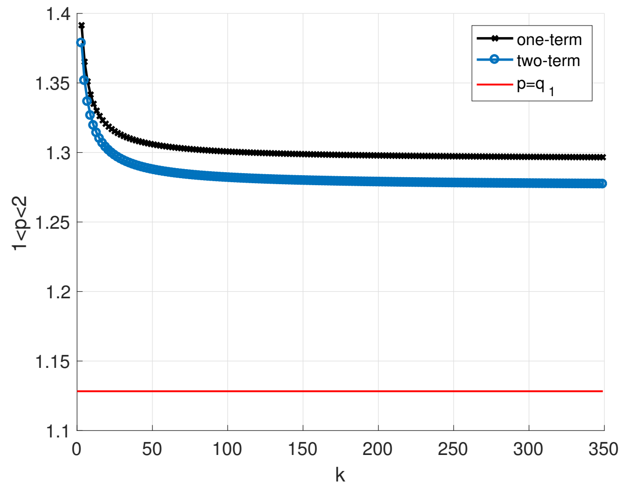

Figure 5.2: Two plots of the points (p,k); p is the root of the equations in (4.14) and (5.7) corresponding to $k \in \mathbb{N}$. Using the improved *one-term* and the *two-term* criteria in the study of bases properties of $\mathfrak{E}_{\cos_p}$, $p \in (1,2)$.

The results of Corollary 5.2 can be further improved by simply using the following inequality

$$\frac{\pi_p \mathcal{M}_p}{\pi^2}(\pi^2 - 8 - 8/81) < b_1(p) + b_9(p).$$

This holds for all $p \in [p_{0,2}, p_{0,1}]$ where $p_{0,2} \approx 1.400566 < p_{0,1}$ under the same conditions in Corollary 5.2.

Corollary 5.3. *Let $3 \le k \equiv_2 1$. Let $p \in (1,2)$ be such that $b_9(p) > 0$ and $|b_3(p)(b_9(p) + b_1(p))| \ge 4|b_9(p)b_1(p)|$. If*

$$\pi_p \mathcal{M}_p < \left(b_1(p) + 2b_9(p) - \sum_{j=3}^{k} |b_j(p)|\right) \frac{\pi^2}{8(\pi^2/8 - \sum_{\substack{j=1 \\ j \equiv_2 1}}^{k} (1/j^2))}, \tag{5.7}$$

then the family $\mathfrak{E}_{\cos_p}$ is a Riesz basis of $L^2(0,1)$.

Proof. For $f(x) = \cos_p(\pi_p x), d = 1$ and $\mathfrak{p} = 3$, we have $\mathcal{F} = \{1,3,9\}$ and $m(z) = b_1(p) + b_3(p)z + b_9(p)z^2$ for $z \in \mathbb{D}$. By virtue of Lemma 5.1, statement (3.23) is satisfied. Then since $|b_3(p)(b_9(p)+b_1(p))| \ge 4|b_9(p)b_1(p)|$, $\omega = b_1(p) + b_9(p) - |b_3(p)|$ (Corollary 3.3*(a)*). In view of statements (5.4) and (5.7) we conclude that (3.22) is satisfied. Hence Corollary 3.2 completes the proof. □

For $k = 321$, statement (5.7) holds for $p \in [p_{0,3}, 2)$ such that $p_{0,3} \approx 1.296718 <$

$p_{0,2}$. This indicates that the family $\mathfrak{E}_{\cos_p}$ forms a Riesz basis in $L^2(0,1)$ for all $p \in [p_{0,3}, 2]$.

5.3.2 Case $p \in (2, \infty)$

Let $p = p_1$ be the solution of the equation

$$\frac{\pi}{p^2 \sin^2(\pi/p)} \left[2 + \frac{\pi^2}{2}(p-2)\right] \left[\left(1 - \frac{1}{2^{p'}}\right) \zeta(p') - 1\right] = 1,$$

which is approximately equal to 2.42865 (see statement (4.19) and Lemma 4.10 for more details).

Lemma 5.3. *There exists $p_{1,1} \in (p_1, 3)$ such that*

$$g(p) := \frac{\pi}{p^2 \sin^2(\pi/p)} \left[2 + \frac{\pi^2}{2}(p-2)\right] \left[\left(1 - \frac{1}{2^{p'}}\right) \zeta(p') - 1 - \frac{1}{9^{p'}}\right] = 1. \tag{5.8}$$

Moreover, for all $p \in [p_1, p_{1,1}]$

$$\frac{\pi}{p^2 \sin^2(\pi/p)} \left[2 + \frac{\pi^2}{2}(p-2)\right] \left[\left(1 - \frac{1}{2^{p'}}\right) \zeta(p') - 1 - \frac{1}{9^{p'}}\right] < 1. \tag{5.9}$$

Proof. In view of the fact that (4.20) holds for $p = p_1$,

$$g(p) = 1 - \frac{\pi}{p^2 \sin^2(\pi/p)} \left[2 + \frac{\pi^2}{2}(p-2)\right] \frac{1}{9^{p'}} < 1 \quad \text{at} \quad p = p_1.$$

On the other hand,

$$\begin{aligned} g(3) &= \frac{\pi}{9 \sin^2(\frac{\pi}{3})} \left[2 + \frac{\pi^2}{2}\right] \left[\left(1 - \frac{1}{2^{\frac{3}{2}}}\right) \zeta\left(\frac{3}{2}\right) - 1 - \frac{1}{9^{3/2}}\right] \\ &> \frac{\pi}{9 \sin^2(\frac{\pi}{3})} \left[2 + \frac{\pi^2}{2}\right] \left[\left(1 - \frac{1}{2^{\frac{3}{2}}}\right) \sqrt{2} \zeta\left(2\right) - 1 - \frac{1}{9^{3/2}}\right] \\ &> 1. \end{aligned}$$

By continuity, there exists $p_{1,1} \in (p_1, 3)$ such that (5.8) holds.

Consider statements (4.15), (4.16) and (4.17). Observe that for any $q \in (1, \infty)$,

$$\begin{aligned}\frac{\zeta'(q)}{\zeta(q)} &< -\frac{\ln 2}{2^q}\left[1 + \frac{1}{2^q} + \frac{1}{4^q} + ...\right] - \frac{\ln 3}{3^q}\left[1 + \frac{1}{3^q} + \frac{1}{9^q} + ...\right] \\ &= -\frac{\ln 2}{2^q - 1} - \frac{\ln 3}{3^q - 1}\end{aligned}$$

and

$$\zeta(q)\left(1 - \frac{1}{2^q}\right) > 1.$$

Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}q}\left[\left(1 - \frac{1}{2^q}\right)\zeta(q) - 1 - \frac{1}{9^q}\right] &= \zeta(q)\left[\frac{\ln 2}{2^q} + \left(1 - \frac{1}{2^q}\right)\frac{\zeta'(q)}{\zeta(q)}\right] + \frac{2\ln 3}{9^q} \\ &< \zeta(q)\left[\left(\frac{1}{2^q} - 1\right)\frac{\ln 3}{3^q - 1} + \frac{1}{\zeta(q)}\frac{2\ln 3}{9^q}\right] \\ &< \zeta(q)\left(1 - \frac{1}{2^q}\right)\ln 3\left[-\frac{1}{3^q - 1} + \frac{2}{9^q}\right] < 0.\end{aligned}$$

This implies that $(1 - 2^{-q})\zeta(q) - 1 - \frac{1}{9^q}$ is decreasing in $q \in (1, \infty)$. Now, p' and $\sin(\pi/p)$ are decreasing functions in $p \in (1, \infty)$ and $p \in (2, \infty)$ respectively. Moreover, $(1 - 1/4)\zeta(2) - 1 - 1/81 > 0$. Then the function

$$\frac{\pi}{\sin^2(\pi/p)}\left[\left(1 - \frac{1}{2^{p'}}\right)\zeta(p') - 1 - \frac{1}{9^{p'}}\right]$$

is increasing in $p \in (2, \infty)$.

We also know that $\frac{1}{p^2}\left(2 + \frac{\pi^2}{2}(p-2)\right)$ is a positive increasing function in $p \in [2, 3]$ (see the proof of Lemma 4.10 for further details).

Consequently, $g(p)$ is an increasing function in $p \in [2, 3]$. This implies that $g(p) < g(p_{1,1}) = 1$ for all $p \in [p_1, p_{1,1}]$, and the result follows. □

Corollary 5.4. *Let $p_{1,1} \in (2, \infty)$ be such that (5.8) holds, $b_9(p) > 0$ and $|b_3(b_9 + b_1)| \geq 4|b_9 b_1|$. Then for any $p \in [p_1, p_{1,1}]$ the family $\mathfrak{E}_{\cos_p}$ is a Riesz basis of $L^2(0, 1)$.*

Proof. Note that statement (3.23) of Corollary 3.3 is fulfilled due to Lemma 5.1. By virtue of (5.4) we have

$$\begin{aligned}\sum_{j\in\mathbb{N}\setminus\{1,9\}}^{\infty}|b_j| &< \frac{2\pi_{p'}}{\pi^2(p-1)}\left[2+\frac{\pi^2}{2}(p-2)\right]\sum_{\substack{j\in\mathbb{N}\setminus\{1,9\}\\ j\equiv_2 1}}^{\infty}\frac{1}{j^{p'}}\\ &= \frac{2\pi_{p'}}{\pi^2(p-1)}\left[2+\frac{\pi^2}{2}(p-2)\right]\left[\left(1-\frac{1}{2^{p'}}\right)\zeta(p')-1-\frac{1}{9^{p'}}\right].\end{aligned}$$

We know that $\sin_p(\cdot)$ is concave on $[0,\pi_p/2]$. Due to Lemma 4.5 we conclude

$$\begin{aligned}b_1(p)+b_9(p) > b_1(p) &= \frac{\pi}{\pi_p}a_1(p) = 4\frac{\pi}{\pi_p}\int_0^{\frac{1}{2}}\sin_p(\pi_p x)\sin(\pi x)\mathrm{d}x\\ &> 4\frac{\pi}{\pi_p}\int_0^{\frac{1}{2}}2x\sin(\pi x)\mathrm{d}x = \frac{8}{\pi\pi_p},\end{aligned}$$

and this is because of the assumption that $b_9(p)>0$.

By virtue of statement (5.9), Lemma 1.2 and Corollary 3.3*(a)* we confirm the assertion claimed. □

Numerically we confirm that the inequalities $|b_3(b_9+b_1)|\geq 4|b_9b_1|$ and $b_9(p)>0$ hold for $p\in(2,4)$; the root $p_{1,1}$ of (5.8) approximately equals to $2.462328\in(p_1,4)$.

We claim that the inequality

$$\frac{2\pi_{p'}}{\pi^2(p-1)}\left[2+\frac{\pi^2}{2}(p-2)\right]\left[\left(1-\frac{1}{2^{p'}}\right)\zeta(p')-1-\frac{1}{9^{p'}}\right] < b_1(p)+b_9(p)$$

under the same conditions of Corollary 5.4, provides an improved threshold $p_{1,2}\approx 2.561986>p_{1,1}$.

Corollary 5.5. *Let $3\leq k\equiv_2 1$. Let $p\in(2,\infty)$ be such that $b_9(p)>0$ and $|b_3(p)(b_9(p)+b_1(p))|\geq 4|b_9(p)b_1(p)|$. If*

$$\frac{2\pi_{p'}[2+\frac{\pi^2}{2}(p-2)]}{\pi^2(p-1)}\Big[\Big(1-\frac{1}{2^{p'}}\Big)\zeta(p')-\sum_{\substack{j=1\\ j\equiv_2 1}}^{k}(1/j^{p'})\Big] < b_1(p)+2b_9(p)-\sum_{j=3}^{k}|b_j(p)|, \tag{5.10}$$

then the family $\mathfrak{E}_{\cos_p}$ is a Riesz basis of $L^2(0,1)$.

Proof. Note that the condition (3.23) is satisfied by Lemma 5.1 for the case $\mathfrak{p}=3$

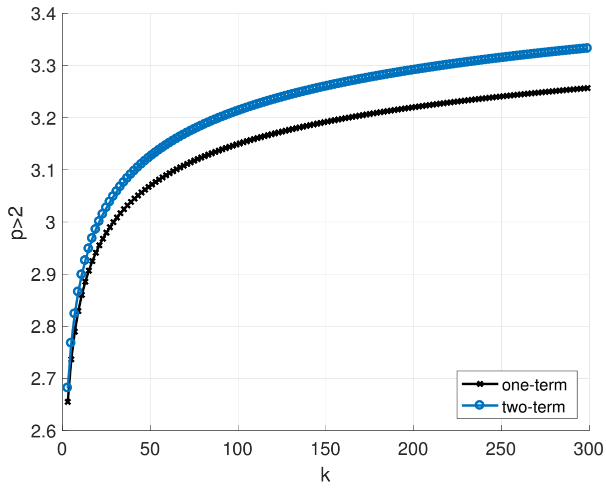

Figure 5.3: The zeros of (4.25) and (5.10), considered as equations in p, against $k \in \mathbb{N}$. Using the improved *one-term* Schauder basis criterion and the *two-term* Riesz basis criterion, respectively, we detect thresholds so that $\mathfrak{E}_{\cos_p}$ forms a basis when $p \in (2, \infty)$.

and $f(x) = \cos_p(\pi_p x)$. According to the proof of Corollary 3.3*(a)*, the inequality $|b_3(p)(b_9(p) + b_1(p))| \geq 4|b_9(p)b_1(p)|$ implies that $\omega = b_1(p) + b_9(p) - |b_3(p)|$. Also, statement (5.4) together with inequality (5.10) yield (3.22). Applying Corollary 3.2, the result follows. $\square$

Consider $k = 321$. Statement (5.10) holds for any $p \in [p_{1,2}, p_{1,3}]$ such that $p_{1,3} \approx 3.339563$.

Remark 5.3. *Observations from Theorem 4.1, Remarks 4.2, 4.4 and Corollaries 5.2, 5.3, 5.4 and 5.5 provide improvements in the thresholds of invertibility of* T *by means of both analytical and numerical approaches. As a consequence, the family* $\mathfrak{E}_{\cos_p}$ *forms a Schauder basis of* $L^r(0,1)$, $r \in (1, \infty)$ *whenever* $p \in [\hat{p}_{0,1}, \hat{p}_{1,1}]$ *and a Riesz basis of* $L^2(0,1)$ *for all* $p \in [p_{0,3}, p_{1,3}]$ *where* $\hat{p}_{0,1} \approx 1.2978$, $\hat{p}_{1,1} \approx 3.2205$, $p_{0,3} \approx 1.296718$ *and* $p_{1,3} \approx 3.339563$.

5.4 Bases properties of $\mathfrak{E}_{\exp_p}$

Let

$$\exp_p(iy) = \cos_p(y) + i \sin_p(y) \qquad \forall y \in \mathbb{R}.$$

By combining Remarks 5.1, 5.3 and Corollary 4.3, it immediately follows that $\mathfrak{E}_{\exp_p} = \{\exp_p(in\pi_p\cdot)\}_{n=-\infty}^{\infty}$ is a Schauder basis of the Banach space $L^r(-1,1)$ for all $p \in [\hat{p}_{0,1}, \hat{p}_{1,1}]$ and $r \in (1,\infty)$, and a Riesz basis of $L^2(0,1)$ for all $p \in [p_{0,3}, p_{1,3}]$.

Indeed, recall that every $f \in L^r(-1,1)$ decomposes as $f = f_{\mathrm{e}} + f_{\mathrm{o}}$ for

$$f_{\mathrm{e}}(x) = \frac{f(x)+f(-x)}{2} \qquad \text{and} \qquad f_{\mathrm{o}}(x) = \frac{f(x)-f(-x)}{2},$$

the even and odd parts of f, respectively. The family $\mathfrak{E}_{\cos_p}$ comprises only even functions, the family $\mathfrak{E}_{\sin_p}$ comprises only odd functions and they are Schauder bases of the corresponding subspaces of $L^r(-1,1)$ for $p \in [\hat{p}_{0,1}, \hat{p}_{1,1}]$ when $r \in (1,\infty)$, and Riesz bases of the corresponding subspaces of $L^2(-1,1)$ for $p \in [p_{0,3}, p_{1,3}]$. This implies that there exist two unique scalar sequences $(\alpha_k)_{k=0}^{\infty}$ and $(\beta_k)_{k=1}^{\infty}$, such that

$$f(\cdot) = \alpha_0 + \sum_{k=1}^{\infty} \alpha_k \cos_p(k\pi_p\cdot) + i\beta_k \sin_p(k\pi_p\cdot)$$

in $L^r(-1,1)$. In order to see this, one expands f_{e} in $\mathfrak{E}_{\cos_p}$ and f_{o} in $\mathfrak{E}_{\sin_p}$, in the corresponding even and odd subspaces.

By letting $c_0 = \alpha_0$,

$$c_k = \frac{\alpha_k + \beta_k}{2} \qquad \text{and} \qquad c_{-k} = \frac{\alpha_k - \beta_k}{2} \qquad \forall k \in \mathbb{N},$$

we get

$$f(\cdot) = \sum_{k=-\infty}^{\infty} c_k \exp_p(ik\pi_p\cdot)$$

in $L^r(-1,1)$. Since there is a 1:1 correspondence between the scalar sequences via

$$\alpha_k = c_k + c_{-k} \qquad \text{and} \qquad \beta_k = c_k - c_{-k},$$

then in fact $(c_k)_{k=-\infty}^{\infty}$ is unique for the given f. Thus, $\mathfrak{E}_{\exp_p}$ satisfies the definitions of Riesz or Schauder bases for the space $L^r(-1,1)$, $r \in (1,\infty)$.

Chapter 6

Schauder Basis Properties of (p, q)-trigonometric Functions and Applications

In this chapter we are interested in finding the (p, q)-values for which the basis properties hold for the set $\mathfrak{E}_{\cos_{p,q}}$. Our results, given in Section 6.4 below, complement the statement provided in [12, Theorem 3.1 and Remark 3.2] which establishes the existence of a $\mathbf{p}_0 \in (1, 2)$, defined as the root of the equation (2.3), such that the family $\mathfrak{E}_{\cos_{p,p'}}$ is a basis in $L^r(0, 1)$ for $p \in [\mathbf{p}_0, 2]$ and $r \in (1, \infty)$ (see Section 2.1.1).

Due to Riemann-Lebesgue Lemma 2.2, the decay properties of the classical Fourier coefficients of a function which is Lebesgue integrable on $(0, 1)$ are known. Here, we investigate the relationship between the decay properties of the Fourier sine coefficients of a function and those of the corresponding coefficients when the classical sine functions are replaced by the $\sin_{p,q}$ functions, where $p, q \in (1, \infty)$.

6.1 Properties of (p, q)-trigonometric functions

Lemma 6.1. *Let $p_1, p_2 \in (1, \infty)$ be such that $p_1 < p_2$. For $q \in (1, \infty)$ fixed, the function*

$$f(x) = \frac{\sin^{-1}_{p_1,q} x}{\sin^{-1}_{p_2,q} x}$$

is strictly increasing in $x \in (0,1)$.

Proof. Set

$$g(x) = \frac{(1-x^q)^{1/p_1}}{(1-x^q)^{1/p_2}}; \tag{6.1}$$

then $g'(x) < 0$. Observe that $f'(x) = \frac{\sin_{p_2,q}^{-1} x - g(x)\sin_{p_1,q}^{-1} x}{\left(\sin_{p_2,q}^{-1} x\right)^2 (1-x^q)^{1/p_1}}$. Let $G(x) = \sin_{p_2,q}^{-1} x - g(x)\sin_{p_1,q}^{-1} x$; then

$$G'(x) = -g'(x)\sin_{p_1,q}^{-1} x > 0.$$

Thus we conclude that $G(x) > \lim_{x\to 0^+} G(x) = 0$. Hence $f'(x) > 0$ for all $x \in (0,1)$. □

Corollary 6.1. *Let $p_1, p_2 \in (1,\infty)$ be such that $p_1 < p_2$. Let $q \in (1,\infty)$ be fixed. Then*

(a) For $x \in (0,1)$, $1 < \frac{\sin_{p_1,q}^{-1} x}{\sin_{p_2,q}^{-1} x} < \frac{\pi_{p_1,q}}{\pi_{p_2,q}}$. Moreover, $\sin_{p_1,q}(\pi_{p_1,q}x) > \sin_{p_2,q}(\pi_{p_2,q}x)$ for all $x \in (0,1/2)$.

(b) For $x \in (0,1/2)$, $\cos_{p_1,q}(\pi_{p_1,q}x) < \cos_{p_2,q}(\pi_{p_2,q}x)$.

Proof. L'Hôpital's Rule yields,

$$\lim_{x\to 0^+} \frac{\sin_{p_1,q}^{-1} x}{\sin_{p_2,q}^{-1} x} = \lim_{x\to 0^+} \frac{(1-x^q)^{-1/p_1}}{(1-x^q)^{-1/p_2}} = 1.$$

According to Lemma 6.1,

$$1 = \lim_{x\to 0^+} \frac{\sin_{p_1,q}^{-1} x}{\sin_{p_2,q}^{-1} x} < \frac{\sin_{p_1,q}^{-1} x}{\sin_{p_2,q}^{-1} x} < \frac{\sin_{p_1,q}^{-1}(1)}{\sin_{p_2,q}^{-1}(1)} = \frac{\pi_{p_1,q}}{\pi_{p_2,q}}.$$

Then *(a)* follows since

$$\frac{1}{\pi_{p_1,q}} \sin_{p_1,q}^{-1} x < \frac{1}{\pi_{p_2,q}} \sin_{p_2,q}^{-1} x.$$

To show *(b)*. We know from Lemma 1.5*(a)* that

$$\sin_{p,q}^{-1} y = \cos_{p,q}^{-1}\left((1-y^q)^{1/p}\right), \qquad \forall y \in [0,1].$$

For $p_1 < p_2$ and $y \in (0, 1)$,

$$(1 - y^q)^{1/p_1} < (1 - y^q)^{1/p_2}. \tag{6.2}$$

This inequality together with part *(a)* yield

$$\frac{1}{\pi_{p_1,q}} \cos^{-1}_{p_1,q} \left((1 - y^q)^{1/p_1}\right) < \frac{1}{\pi_{p_2,q}} \cos^{-1}_{p_2,q} \left((1 - y^q)^{1/p_2}\right).$$

Hence, because of (6.2) and the first statement in *(a)*, the inverse to the function $\frac{1}{\pi_{p,q}} \cos^{-1}_{p,q} \left((1 - y^q)^{1/p}\right)$ strictly increases in p whenever q is fixed. □

Corollary 6.2. *Let $q_1, q_2 \in (1, \infty)$ be such that $q_1 < q_2$. Let $p \in (1, \infty)$ be fixed. Then for $x \in (0, 1/2)$,*

(a) $[\sin_{p,q_1}(\pi_{p,q_1} x)]^{q_1 - 1} > [\sin_{p,q_2}(\pi_{p,q_2} x)]^{q_2 - 1}$.

(b) $\cos_{p,q_1}(\pi_{p,q_1} x) < \cos_{p,q_2}(\pi_{p,q_2} x)$.

Proof. By Lemma 1.5*(c)* and the change of variable to $x = (1 - y)/2$, we see that

$$\sin_{p,q}(\pi_{p,q} x) = [\cos_{q',p'}(\pi_{q',p'}(1/2 - x))]^{1/(q-1)}.$$

Observe that q' decreases as q increases. According to part *(b)* of Corollary 6.1, $\cos_{q_1',p'}(\pi_{q_1',p'}(1/2 - x)) > \cos_{q_2',p'}(\pi_{q_2',p'}(1/2 - x))$, and the first statement follows.

On the other hand, for $x \in [0, 1/2]$ by Lemma 1.5

$$\cos_{p,q}(\pi_{p,q} x) = [\sin_{q',p'}(\pi_{q'p'}(1/2 - x))]^{1/(p-1)}.$$

Since q' is decreasing in q, by virtue of part *(a)* of Corollary 6.1 the proof is complete. □

The following two lemmas are generalisations of Lemmas 4.3 and 4.4 to the case of $\sin_{p,q}$ and $\cos_{p,q}$ functions. They demonstrate some properties of functions that are employed in Sections 6.2.1 and 6.2.2 below. The proofs are similar to those of Chapter 4 and are therefore omitted.

6.1.1 The case $p \in (1,2)$ and $q \in (1,\infty)$

For $p \in (1,2)$ and $q \in (1,\infty)$, let $\mathrm{u}_{p,q} : [0, \frac{1}{2}] \longrightarrow \mathbb{R}$ be given by

$$\mathrm{u}_{p,q}(x) := -\frac{q}{p}\left[\sin_{p,q}(\pi_{p,q}x)\right]^{q-1}\left[\cos_{p,q}(\pi_{p,q}x)\right]^{2-p}.$$

We write

$$\mathcal{C}_{p,q} := \frac{q}{p}\left[\frac{q(2-p)}{2q-p}\right]^{\frac{2-p}{p}}\left[\frac{p(q-1)}{2q-p}\right]^{\frac{q-1}{q}}. \tag{6.3}$$

Observe that

$$\mathrm{u}'_{p,q}(x) = -\frac{q}{p}\pi_{p,q}\,\mathrm{u}_{p,q}(x)\frac{\left[\cos_{p,q}(\pi_{p,q}x)\right]^{1-p}}{\sin_{p,q}(\pi_{p,q}x)}\left[2-p-\left(2-\frac{p}{q}\right)\left[\cos_{p,q}(\pi_{p,q}x)\right]^{p}\right].$$

The functions $\mathrm{u}_{p,q}$ have the following properties:

Lemma 6.2. *Let $p \in (1,2)$ and $q \in (1,\infty)$. Then*

(a) The function $\mathrm{u}_{p,q}$ is non-positive. Moreover, $\mathrm{u}_{p,q}(x) = 0$ if and only if $x = 0$ or $x = 1/2$.

(b) $\mathrm{u}_{p,q}(x) = -\mathcal{C}_{p,q}$ for $x \in [0, \frac{1}{2}]$ if and only if $x = m_{p,q} \in (0, \frac{1}{2})$, where $m_{p,q}$ is the unique point such that $[\cos_{p,q}(\pi_{p,q}m_{p,q})]^p = \frac{q(2-p)}{2q-p}$.

(c) $\mathrm{u}_{p,q} : [0, m_{p,q}] \longrightarrow [-\mathcal{C}_{p,q}, 0]$ is decreasing.

(d) $\mathrm{u}_{p,q} : [m_{p,q}, \frac{1}{2}] \longrightarrow [-\mathcal{C}_{p,q}, 0]$ is increasing.

(e) $\min_{x\in[0,\frac{1}{2}]} \mathrm{u}_{p,q}(x) = -\mathcal{C}_{p,q}$.

As a result of parts *(c)* and *(d)* of Lemma 6.2, there exist $\mathrm{w}_{1,p,q} : [-\mathcal{C}_{p,q}, 0] \longrightarrow [0, m_{p,q}]$ and $\mathrm{w}_{2,p,q} : [-\mathcal{C}_{p,q}, 0] \longrightarrow [m_{p,q}, \frac{1}{2}]$, such that

$$\mathrm{u}_{p,q}(\mathrm{w}_{k,p,q}(x)) = x \qquad \forall x \in [-\mathcal{C}_{p,q}, 0], (k = 1, 2).$$

6.1.2 The case $p \in (2,\infty)$ and $q \in (1,\infty)$

For $p \in (2,\infty)$ and $q \in (1,\infty)$, let $\mathrm{v}_{p,q} : (0, \frac{1}{2}] \longrightarrow [0,\infty)$ be given by

$$\mathrm{v}_{p,q}(x) := (p'-1)\left[\sin_{q',p'}(\pi_{q',p'}x)\right]^{p'-2}\cos_{q',p'}(\pi_{q',p'}x).$$

Notice that

$$\frac{\mathrm{v}_{p,q}'(x)}{\mathrm{v}_{p,q}(x)} = \frac{p'\pi_{q',p'}\left[\cos_{q',p'}(\pi_{q',p'}x)\right]^{1-q'}}{q'\sin_{q',p'}(\pi_{q',p'}x)}\left[\cos_{q',p'}^{q'}(\pi_{q',p'}x)\left(\frac{q'(p'-2)}{p'}+1\right)-1\right].$$

The functions $\mathrm{v}_{p,q}$ enjoy the following properties:

Lemma 6.3. *Let $p \in (2,\infty)$ and $q \in (1,\infty)$. Then*

(a) $\mathrm{v}_{p,q}$ *is decreasing in* $(0,1/2]$ *with* $\lim\limits_{x\to 0^+} \mathrm{v}_{p,q}(x) = +\infty$ *and* $\mathrm{v}_{p,q}(\frac{1}{2}) = 0$.

(b) $\lim\limits_{x\to 0^+} x\,\mathrm{v}_{p,q}(x) = 0$.

(c) $\lim\limits_{x\to 0^+} \mathrm{v}_{p,q}'(x) = -\infty$; $\lim\limits_{x\to \frac{1}{2}^-} \mathrm{v}_{p,q}'(x) = -\infty$ *whenever* $q \in (1,2)$ *and* $\lim\limits_{x\to \frac{1}{2}^-} \mathrm{v}_{p}'(x) = 0$ *whenever* $q \in (2,\infty)$.

Consequently $\mathrm{v}_{p,q}$ has an inverse $\mathrm{z}_{p,q} : [0,\infty) \to (0,\frac{1}{2}]$ which is decreasing in $[0,\infty)$. Moreover, $\mathrm{z}_{p,q}(0) = \frac{1}{2}$ and $\lim\limits_{x\to\infty} \mathrm{z}_{p,q}(x) = 0$.

6.2 The Fourier coefficients of the (p,q)-trigonometric functions

Let $\eta_j(p,q)$ and $\tau_j(p,q)$ be the Fourier coefficients of $\cos_{p,q}(\pi_{p,q}x)$ and $\sin_{p,q}(\pi_{p,q}x)$, respectively. Since $\sin_{p,q}$ is even and $\cos_{p,q}$ is odd around $\frac{1}{2}$, $\eta_j(p,q) = \tau_j(p,q) = 0$ for all $j \equiv_2 0$ and $p,q \in (1,\infty)$. Then

$$\eta_j := \eta_j(p,q) = \begin{cases} 0 & j \equiv_2 0 \\ 2\int_0^1 \cos_{p,q}(\pi_{p,q}x)\cos(j\pi x)\mathrm{d}x & j \equiv_2 1. \end{cases} \tag{6.4}$$

and

$$\tau_j := \tau_j(p,q) = \begin{cases} 0 & j \equiv_2 0 \\ 2\int_0^1 \sin_{p,q}(\pi_{p,q}x)\sin(j\pi x)\mathrm{d}x & j \equiv_2 1. \end{cases} \tag{6.5}$$

Observe that for any $j \in \mathbb{N}$ (see Lemma 4.5 for the case $p = q$),

$$\eta_j(p,q) = \frac{j\pi}{\pi_{p,q}} \tau_j(p,q). \tag{6.6}$$

We now find upper bound estimates of $|\eta_j(p,q)|$ and $|\tau_j(p,q)|$ in terms of the parameters $(p,q) \in (1,\infty)^2$.

6.2.1 The case $p \in (1,2)$ and $q \in (1,\infty)$

Lemma 6.4. *Let $\mathcal{C}_{p,q} > 0$ be given by* (6.3). *Then*

$$|\eta_j(p,q)| < \frac{8\pi_{p,q}}{j^2\pi^2} \mathcal{C}_{p,q}, \qquad \forall j \geq 1. \tag{6.7}$$

Proof. Integration by parts ensures that

$$\begin{aligned}
\eta_j(p,q) &= 4 \int_0^{\frac{1}{2}} \cos_{p,q}(\pi_{p,q} x) \cos(j\pi x) \mathrm{d}x \\
&= \left[\frac{4}{j\pi} \cos_{p,q}(\pi_{p,q} x) \sin(j\pi x) \right]_0^{\frac{1}{2}} - \frac{4\pi_{p,q}}{j\pi} \int_0^{\frac{1}{2}} \mathrm{u}_{p,q}(x) \sin(j\pi x) \mathrm{d}x \\
&= \left[\frac{4\pi_{p,q}}{j^2\pi^2} \mathrm{u}_{p,q}(x) \cos(j\pi x) \right]_0^{\frac{1}{2}} - \frac{4\pi_{p,q}}{j^2\pi^2} \int_0^{\frac{1}{2}} \mathrm{u}'_{p,q}(x) \cos(j\pi x) \mathrm{d}x.
\end{aligned}$$

Thus by Lemma 6.2*(a)*, it follows that the boundary terms in the last equality vanish.

$$\begin{aligned}
\eta_j(p,q) &= -\frac{4\pi_{p,q}}{j^2\pi^2} \Big(\int_0^{m_{p,q}} \mathrm{u}'_{p,q}(x) \cos(j\pi x) \mathrm{d}x + \int_{m_{p,q}}^{\frac{1}{2}} \mathrm{u}'_{p,q}(x) \cos(j\pi x) \mathrm{d}x \Big) \\
&= -\frac{4\pi_{p,q}}{j^2\pi^2} \Big(\int_0^{-\mathcal{C}_{p,q}} \cos(j\pi\, \mathrm{w}_{1,p,q}(s)) ds + \int_{-\mathcal{C}_{p,q}}^0 \cos(j\pi\, \mathrm{w}_{2,p,q}(s)) \mathrm{d}s \Big).
\end{aligned}$$

Consequently

$$\begin{aligned}
|\eta_j(p,q)| &\leq \frac{4\pi_{p,q}}{j^2\pi^2} \Big[\int_{-\mathcal{C}_{p,q}}^0 |\cos(j\pi\, \mathrm{w}_{1,p,q}(s))| ds + \int_{-\mathcal{C}_{p,q}}^0 |\cos(j\pi\, \mathrm{w}_{2,p,q}(s))| \mathrm{d}s \Big] \\
&< \frac{8\pi_{p,q}}{j^2\pi^2} \mathcal{C}_{p,q}.
\end{aligned}$$

$\square$

Moreover, using identity (6.6) we conclude

$$|\tau_j(p,q)| < \frac{8\pi_{p,q}^2}{j^3\pi^3}\mathcal{C}_{p,q}, \qquad \forall j \geq 1. \tag{6.8}$$

6.2.2 The case $p \in (2,\infty)$ and $q \in (1,\infty)$

By Lemma 1.5(c),

$$\eta_j(p,q) = 4\int_0^{\frac{1}{2}} \left[\sin_{q',p'}\left(\pi_{q',p'}(1/2-x)\right)\right]^{1/(p-1)} \cos(j\pi x)\mathrm{d}x.$$

Notice that, $\cos(j\pi(\frac{1}{2}-t)) = (-1)^{\frac{j-1}{2}}\sin(j\pi t)$ for $j \equiv_2 1$. Changing variables to $t = \frac{1}{2} - x$ gives

$$\eta_j = (-1)^{\frac{j-1}{2}} 4\int_0^{\frac{1}{2}} [\sin_{q',p'}(\pi_{q',p'}t)]^{\frac{1}{p-1}} \sin(j\pi t)\mathrm{d}t.$$

Now use Lemma 6.3 and integration by parts twice to obtain,

$$\begin{aligned}\eta_j &= (-1)^{\frac{j-1}{2}}\frac{4\pi_{q',p'}}{j\pi}\int_0^{\frac{1}{2}} \mathrm{v}_{p,q}(t)\cos(j\pi t)\mathrm{d}t = (-1)^{\frac{j+1}{2}}\frac{4\pi_{q',p'}}{j^2\pi^2}\int_0^{\frac{1}{2}} \mathrm{v}'_{p,q}(t)\sin(j\pi t)\mathrm{d}t \\ &= (-1)^{\frac{j+3}{2}}\frac{4\pi_{q',p'}}{j^2\pi^2}\int_0^{\infty} \sin\left(j\pi\,\mathrm{z}_{p,q}(y)\right)\mathrm{d}y.\end{aligned}$$

Lemma 6.5. *Let $p \in (2,\infty)$ and $q \in (1,\infty)$. Then*

$$|\eta_j(p,q)| < \frac{4\pi_{q',p'}}{\pi^2}\left[p'-1+\frac{\pi}{2}(2-p')\right] j^{-p'}, \qquad \forall j \geq 1. \tag{6.9}$$

Proof. Since $p > 2$, then $p' \in (1,2)$ and $p'-1 \in (0,1)$. By Lemma 1.6, $\sin_{q',p'}(\pi_{q',p'}t) \geq 2t$ for any $t \in (0,\frac{1}{2}]$. Then by Lemma 6.3, we have $\mathrm{v}_{p,q}(t) < (p'-1)(2t)^{p'-2}$ and so

$$\mathrm{z}_{p,q}(y) < \frac{1}{2}\left(\frac{y}{p'-1}\right)^{\frac{1}{p'-2}} =: \mathrm{r}_{p,q}(y), \qquad \forall y \in [p'-1,\infty).$$

With $\xi(j) := (p'-1)/j^{p'-2}$, we see that $\mathrm{r}_{p,q}(\xi(j)) = \frac{1}{2j} \leq \frac{1}{2}$ for all $j \geq 1$.

Set

$$\mathrm{J}_1 = \int_0^{\xi(j)} \mathrm{d}x = \xi(j) \qquad \text{and} \qquad \mathrm{J}_2 = \int_{\xi(j)}^{\infty} \sin\left(j\pi\, \mathrm{r}_{p,q}(y)\right) \mathrm{d}y.$$

For $y \in [\xi(j), \infty)$ and since $z_{p,q}(\cdot)$ is decreasing on $[0,\infty)$, we obtain $0 < j\pi\, \mathrm{z}_{p,q}(y) \leq j\pi\, \mathrm{z}_{p,q}(\xi(j)) < j\pi\, \mathrm{r}_{p,q}(\xi(j)) = \frac{\pi}{2}$. Hence,

$$\int_{\xi(j)}^{\infty} \sin\left(j\pi\, \mathrm{z}_{p,q}(y)\right) \mathrm{d}y < \mathrm{J}_2 \implies |\eta_j(p,q)| < \frac{4\pi_{q',p'}}{j^2\pi^2}(\mathrm{J}_1 + \mathrm{J}_2).$$

Changing variables to $t = j\pi\, \mathrm{r}_{p,q}(y)$ and then using the fact that $\sup\limits_{0<\theta\leq\frac{\pi}{2}} \dfrac{\sin\theta}{\theta} = 1$ yields

$$\begin{aligned}\mathrm{J}_2 &= \int_0^{\frac{\pi}{2}} (p'-1)(2-p')\left(\frac{2}{j\pi}\right)^{p'-2} t^{p'-2}\frac{\sin(t)}{t}\mathrm{d}t \\ &\leq (p'-1)(2-p')\left(\frac{2}{j\pi}\right)^{p'-2} \int_0^{\frac{\pi}{2}} t^{p'-2}\mathrm{d}t = \frac{\pi}{2}(2-p')j^{2-p'},\end{aligned}$$

and the result follows. □

In view of statement (6.6) and Lemma 1.2,

$$|\tau_j(p,q)| < \frac{4q\pi_{p,q}^2}{p'\pi^3}\left[p' - 1 + \frac{\pi}{2}(2-p')\right] j^{-p'-1}, \qquad \forall j \geq 1. \tag{6.10}$$

From statements (6.7) and (6.9) we conclude that $\sum_{j=1}^{\infty} |\eta_j(p,q)| < \infty$. Thus Proposition 4.1 is fulfilled for $\eta_j(p,q)$ which guarantees the boundedness of the change of coordinates map.

6.3 The decay of the (p,q)-sine Fourier coefficients

This section offers a comparison between the decay rates of the sine Fourier coefficients of a function f in $L^r(0,1)$ and those of the Fourier expansion of f expressed in terms of the basis formed by the $\sin_{p,q}$ functions. The comparison is conducted in both Lebesgue and Lorentz sequence spaces, and is mainly based on the speed of decay of the sine Fourier coefficients τ_j, and also on (2.2) which sufficiently ensures the property of the set $\mathfrak{E}_{\sin_{p,q}}$ being a basis. Theorem 6.1 constitutes improvements of those stated in Lemma 2.10.

Theorem 6.1. *Let $p,q \in (1,\infty)$ be such that (2.2) holds. Let $f \in L^r(0,1)$ have the representation (2.4) and consider γ given by (6.20). Suppose that the sequences $(|\mathbf{a}_k|)_{k\in\mathbb{N}}$ and $(|\mathbf{b}_k|)_{k\in\mathbb{N}}$ are non-increasing and let $\alpha \in (0,\gamma] \subseteq (0,3]$. Then, $|\mathbf{b}_k| \lesssim k^{-\alpha}$ if and only if $|\mathbf{a}_k| \lesssim k^{-\alpha}$.*

Proof. Suppose that $|\mathbf{b}_k| \lesssim k^{-\alpha}$. By Lemma 2.9, $\mathbf{a}_k = \sum_{\substack{m,n\in\mathbb{N}\\ mn=k}} \mathbf{b}_n \tau_m$, which we write as $\mathbf{a}_k = \sum_{n|k} \mathbf{b}_n \tau_{k/n}$ $(k \in \mathbb{N})$, where the summation is over all $n \in \mathbb{N}$ which divide k, written $n|k$. As $(|\mathbf{a}_k|)_{k\in\mathbb{N}}$ is non-increasing, it is sufficient to investigate the decay when $k = 2^l$, $l \in \mathbb{N}$, and in this case

$$\mathbf{a}_{2^l} = \sum_{j=1}^{l} \mathbf{b}_{2^j} \tau_{2^{l-j}}.$$

For the case $p \in (1,2)$, we have $|\tau_j(p,q)| \lesssim j^{-3}$. Then

$$|\mathbf{a}_{2^l}| \lesssim \sum_{j=1}^{l} 2^{-\alpha j} 2^{-3(l-j)} \lesssim 2^{-\alpha l}.$$

Moreover for $p \in (2,\infty)$, $|\tau_j(p,q)|$ decays like $j^{-p'-1}$. From this we conclude that

$$|\mathbf{a}_{2^l}| \lesssim \sum_{j=1}^{l} 2^{-\alpha j} 2^{-(p'+1)(l-j)} \lesssim 2^{-\alpha l},$$

given that $p'+1 \in (2,3)$, and the assertion claimed follows.

The proof of the other implication follows similarly by using Lemma 2.8. $\square$

Note that the theorem shows that the k-th $\sin_{p,q}$ Fourier coefficient decays at the same rate α as the k-th sine Fourier coefficient of the function $f \in L^r(0,1)$ for $\alpha \in (0,3]$ improving that of Lemma 2.10. This holds if, $p'/q < 4/(\pi^2 - 8)$ for $p,q \in (1,\infty)$ or $p = q \in [\tilde{p}_1, \infty)$.

The following considers the study in the context of Lorentz sequence spaces under the same monotonicity assumption, but at the expense of a weaker conclusion, i.e.

the rate of the decay experiences a loss of sharpness (see, Proposition 6.1 below).

Definition 6.1. *[25, Definition 1.b.7, pp. 33] Let $\mathbf{c} = (c_k)_{k\in\mathbb{N}}$ be a null sequence of complex numbers. Denote by $c^* = (c_k^*)_{k\in\mathbb{N}} \subset \mathbb{R}^+$ the non-increasing rearrangement of $(c_k)_{k\in\mathbb{N}}$ given as*

$$c_1^* := \max_{j\in\mathbb{N}} |c_j|, \quad c_k^* := \max_{|I_k|=k} \left(\sum_{j\in I_k} |c_j|\right) - \sum_{j=1}^{k-1} c_j^*, \quad \textit{for} \quad k > 1.$$

Note that $c_1^* \geq c_2^* \geq \ldots$ and the sets $\{|c_k|\}_k$ and $\{|c_k^*|\}_k$ are the same, and the elements are occurring with the same multiplicity. By definition there exists a bijective map $\mathcal{Q} : \mathbb{N} \to \mathbb{N}$ such that $c_k^* = |c_{\mathcal{Q}(k)}|$ for all $k \in \mathbb{N}$.

Definition 6.2. *[25] Let $u \in (0, \infty)$ and $v \in (0, \infty]$. Define the Lorentz sequence space $l_{u,v}$ as the space of all sequences $\mathbf{c} = (\mathbf{c}_k)_{k\in\mathbb{N}}$ such that*

$$\|\mathbf{c}\|_{u,v}^v = \sum_{k=1}^{\infty} |\mathbf{c}_k^*|^v k^{\frac{v}{u}-1} < \infty, \qquad v < \infty.$$

For $v = \infty$. The Lorentz sequence space $l_{u,\infty}$ is defined as the space of all sequences $\mathbf{c} = (c_k)_{k\in\mathbb{N}}$ such that $\|\mathbf{c}\|_{u,\infty} = \sup_{k\in\mathbb{N}} k^{1/u}\mathbf{c}_k^* < \infty$.

Without loss of generality, if we assume that $\|\mathbf{c}\|_{u,\infty} = 1$, then $\mathbf{c}_k^* \leq k^{-1/u}$ for any $k \in \mathbb{N}$. Moreover, it is known that $l_{u,v} \subset l_{u,\infty}$ for any $u, v \in (0, \infty)$ [25, Proposition 1.c.10, pp. 52].

Proposition 6.1. *Let $p, q \in (1, \infty)$ be such that (2.2) holds. Let $f \in L^r(0, 1)$ have the representation (2.4). Assume that the sequences $(|\mathbf{a}_k|)_{k\in\mathbb{N}}$ and $(|\mathbf{b}_k|)_{k\in\mathbb{N}}$ are non-increasing. Let $u, v \in (1, \infty)$ be such that $u < v$ and $\mathbf{b} = (\mathbf{b}_k) \in l_{u,v}$. Then there exists $t \in (1, \infty)$ such that, for $t > uv/(v - u)$ the sequence $\mathbf{a} = (\mathbf{a}_n) \in l_{t,s}$ for all $s \in (1, \infty)$.*

Proof. Since the sequences $(|\mathbf{a}_k|)_{k\in\mathbb{N}}$ and $(|\mathbf{b}_k|)_{k\in\mathbb{N}}$ are non-increasing, $\mathbf{a}_k^* = |\mathbf{a}_k|$ and $\mathbf{b}_k^* = |\mathbf{b}_k|$ for all $k \in \mathbb{N}$, respectively. Let $d(k)$ be the number of divisors of k, including 1 and k: $d(k) = \sum_{m|k} 1$; and put $\sigma_\beta(k) = \sum_{m|k} m^\beta$, so that $\sigma_0(k) = d(k)$.

To prove that $\mathbf{a} \in l_{t,s}$, we need to show that $\sum_{k=1}^{\infty} |\mathbf{a}_k|^s k^{s/t-1} < \infty$.

For $t,s\in(1,\infty)$. According to (2.2), (2.5) and (6.20) we get

$$\sum_{k=1}^{\infty}|\mathbf{a}_k|^s k^{s/t-1}=\sum_{k=1}^{\infty}k^{s/t-1}\left|\sum\nolimits_{n|k}\mathbf{b}_n\tau_{k/n}\right|^s\lesssim\sum_{k=1}^{\infty}k^{s/t-1-\gamma s}\left|\sum\nolimits_{n|k}\mathbf{b}_n n^{\gamma}\right|^s.$$

By Hölder's inequality,

$$\left|\sum\nolimits_{n|k}\mathbf{b}_n n^{\gamma}\right|=\left|\sum\nolimits_{n|k}\mathbf{b}_n n^{1/u-1/v}n^{\gamma-1/u+1/v}\right|\lesssim\ \|\mathbf{b}\|_{u,v}\,\sigma^{1/v'}_{(\gamma-1/u+1/v)v'}(k)$$

It is known that, given any $\epsilon>0$ there exists $k_0(\epsilon)\in\mathbb{N}$ such that for $k\geq k_0(\epsilon)$ (see, [39, pp. 85-86])

$$\left|\sum\nolimits_{n|k}\mathbf{b}_n n^{\gamma}\right|\lesssim\|\mathbf{b}\|_{u,v}\,k^{\gamma-1/u+1/v+\varepsilon}.$$

Hence,

$$\sum_{k=1}^{\infty}|\mathbf{a}_k|^s k^{s/t-1}\lesssim\|\mathbf{b}\|_{u,v}^s\sum_{k=1}^{\infty}k^{s/t-1-\gamma s+s(\gamma-1/u+1/v)+\varepsilon}<\infty$$

if

$$\frac{s}{t}-1-\gamma s+s\left(\gamma-\frac{1}{u}+\frac{1}{v}\right)<-1\iff\frac{1}{t}+\frac{1}{v}<\frac{1}{u}.\tag{6.11}$$

The result follows. □

In the special case when $\mathbf{b}\in l_{v',v}$ such that $v\in(2,\infty)$, condition (6.11) becomes $1/t+2/v<1$. Thus if $v\in(2,\infty)$ and $\mathbf{b}=(\mathbf{b}_k)\in l_{v',v}$, then $\mathbf{a}\in l_{t,s}$ for all $t>v/(v-2)$ and $s\in(1,\infty)$.

For $u\in(1,2)$ and $\mathbf{b}\in l_{u,u'}$ the same condition takes the form $1/t-2/u<-1$. This implies the following: If $u\in(1,2)$ and $\mathbf{b}=(\mathbf{b}_k)\in l_{u,u'}$, $\mathbf{a}\in l_{t,s}$ for all $t>u/(2-u)$ and $s\in(1,\infty)$.

6.4 Bases properties of $\mathfrak{E}_{\cos_{p,q}}$

Following the same argument as in Section 4.3. We employ the *one-term* criterion (see Theorem 3.1 and Remark 3.2 for further details). If

$$\sum_{j=3}^{\infty} |\eta_j(p,q)| < |\eta_1(p,q)|, \tag{6.12}$$

the set $\mathfrak{E}_{\cos_{p,q}}$ is a Schauder basis of $L^r(0,1)$ for all $r \in (1,\infty)$.

One of our main results is that:

Theorem 6.2. *There exist $\mathbf{p_1} < 2$ and $\mathbf{p_2} > 2$ such that for any $p \in [\mathbf{p_1}, \mathbf{p_2}]$, the set $\mathfrak{E}_{\cos_{p,p'}}$ is a Schauder basis in the Banach space $L^r(0,1), r \in (1,\infty)$.*

The values $\mathbf{p_1}$ and $\mathbf{p_2}$ are unique roots of equations involving p obtained by employing (6.12) in the two different cases $p \in (1,2)$ and $p \in (2,\infty)$, respectively (see Corollaries 6.3 and 6.4 below). Numerically it is shown that $\mathbf{p_1} \approx 1.487807$ and $\mathbf{p_2} \approx 2.526402$. This theorem improves the results obtained in [12] which were briefly described in Section 2.1.1 and Table 1.2.

In the following section we employ the criterion stated in (6.12) and identify the thresholds in the parameters p and q for which the family $\mathfrak{E}_{\cos_{p,q}}$ is guaranteed to form a basis in the Banach space $L^r(0,1)$. In particular we are interested in the case when $q = p'$.

We distinguish the two cases:

6.4.1 The case $p \in (1,2)$ and $q \in (1,\infty)$

Theorem 6.3. *The sequence $\mathfrak{E}_{\cos_{p,q}}$ forms a Schauder basis in $L^r(0,1)$, $r \in (1,\infty)$ whenever*

$$\pi_{p,q}^2 \mathcal{C}_{p,q} < \frac{8\pi}{\pi^2 - 8}.$$

Proof. By (6.7),

$$\sum_{j=1}^{\infty}|\eta_{2j+1}(p,q)| < \frac{8\pi_{p,q}}{\pi^2}\mathcal{C}_{p,q}\sum_{j=1}^{\infty}\frac{1}{(2j+1)^2} = \frac{8\pi_{p,q}}{\pi^2}\mathcal{C}_{p,q}\left(\frac{\pi^2}{8}-1\right).$$

By Lemma 1.6, observe that

$$\tau_1(p,q) = 4\int_0^{\frac{1}{2}}\sin_{p,q}(\pi_{p,q}x)\sin(\pi x)\mathrm{d}x \geq 4\int_0^{\frac{1}{2}} 2x\sin(\pi x)\mathrm{d}x = \frac{8}{\pi^2}.$$

Then in view of (6.6), we have

$$|\eta_1(p,q)| = \frac{\pi}{\pi_{p,q}}|\tau_1(p,q)| \geq \frac{8}{\pi\pi_{p,q}}.$$

Thus

$$\sum_{j=1}^{\infty}|\eta_{2j+1}(p,q)| < \frac{8\pi_{p,q}}{\pi^2}\mathcal{C}_{p,q}\left(\frac{\pi^2}{8}-1\right) < \frac{8}{\pi\pi_{p,q}} \leq |\eta_1(p,q)|,$$

which guarantees (6.12). Hence the claimed assertion is complete. □

6.4.1.1 The case $q = p'$

Let $p = \mathbf{p_0} \in (1,2)$ be such that (2.3) (see [12], Theorem 3.1 and Remark 3.2).

Substituting in (6.3),

$$\mathcal{C}_{p,p'} = \frac{1}{p-1}\left[\frac{2-p}{3-p}\right]^{\frac{2-p}{p}}\left[\frac{1}{3-p}\right]^{\frac{1}{p}}.$$

Lemma 6.6. *There exists $\mathbf{p_1} \in (1,\mathbf{p_0})$ such that*

$$\pi^2_{\mathbf{p_1},\mathbf{p_1'}}\mathcal{C}_{\mathbf{p_1},\mathbf{p_1'}} = \frac{8\pi}{\pi^2-8}. \tag{6.13}$$

Moreover, for all $p \in [\mathbf{p_1},\mathbf{p_0}]$,

$$\pi^2_{p,p'}\mathcal{C}_{p,p'} \leq \frac{8\pi}{\pi^2-8}. \tag{6.14}$$

Proof. Note that $2 \leq \pi_{p,p'} \leq 4$ whenever $p \in (1,\infty)$ by Lemma 1.4. For $p \to 1^+$,

$p' \to +\infty$, and for $p=2$, $p'=2$. Observe that,

$$\lim_{p\to 1^+} \pi^2_{p,p'}\mathcal{C}_{p,p'} = \infty \qquad \text{and} \qquad \lim_{p\to 2^-} \pi^2_{p,p'}\mathcal{C}_{p,p'} = \pi^2 < \frac{8\pi}{\pi^2-8}.$$

Then by the Intermediate Value Theorem there exists $\mathbf{p_1} \in (1,2)$ such that the equation (6.13) holds. Moreover, according to (2.3)

$$\pi^2_{\mathbf{p_0},\mathbf{p_0'}}\mathcal{C}_{\mathbf{p_0},\mathbf{p_0'}} = \frac{8\pi}{\pi^2-8}\mathcal{C}_{\mathbf{p_0},\mathbf{p_0'}}(\mathbf{p_0}-1).$$

Then, from the expression of $\mathcal{C}_{p,p'}$ and the fact that its last two factors are exactly less than 1 when $p\in(1,2)$, we conclude that

$$\pi^2_{\mathbf{p_0},\mathbf{p_0'}}\mathcal{C}_{\mathbf{p_0},\mathbf{p_0'}} < \frac{8\pi}{\pi^2-8} = \pi^2_{\mathbf{p_1},\mathbf{p_1'}}\mathcal{C}_{\mathbf{p_1},\mathbf{p_1'}}.$$

Hence $\mathbf{p_1} < \mathbf{p_0} < 2$.

Now, firstly note that

$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{2-p}{3-p}\right)^{\frac{2-p}{p}} = \left(\frac{2-p}{3-p}\right)^{\frac{2-p}{p}}\left[\frac{-2}{p^2}\ln\frac{2-p}{3-p} - \frac{1}{p(3-p)}\right]$$

and

$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{1}{3-p}\right)^{\frac{1}{p}} = \left(\frac{1}{3-p}\right)^{\frac{1}{p}}\left[\frac{1}{p^2}\ln(3-p) + \frac{1}{p(3-p)}\right].$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}p}\mathcal{C}_{p,p'} = \mathcal{C}_{p,p'}\left[\frac{1}{p^2}\ln\frac{(3-p)^3}{(2-p)^2} - \frac{1}{p-1}\right] < 0$$

for any $p\in(1,9/5)$. Hence $\mathcal{C}_{p,p'}$ is decreasing in this interval.

Observe that both $\mathcal{C}_{p,p'}$ and $\pi_{p,p'}$ are positive functions. The function $\pi_{q',q}$ is an increasing function in q and since $q=p'$ is decreasing in p, then $\pi_{p,p'}$ decreases in $p\in(1,\infty)$ (see, (1.9) and (1.10)). Therefore $\pi^2_{p,p'}\mathcal{C}_{p,p'}$ is decreasing in $p\in(1,9/5)$. Hence for any $p\in(\mathbf{p_1},\mathbf{p_0}]$,

$$\pi^2_{p,p'}\mathcal{C}_{p,p'} < \pi^2_{\mathbf{p_1},\mathbf{p_1'}}\mathcal{C}_{\mathbf{p_1},\mathbf{p_1'}} = \frac{8\pi}{\pi^2-8}$$

and the claimed assertion is complete. $\square$

Corollary 6.3. *Let* $\mathbf{p_1} \in (1,2)$ *be such that* (6.13) *holds. Then for any* $p \in [\mathbf{p_1}, \mathbf{p_0}]$ *the sequence* $\mathfrak{E}_{\cos_{p,p'}}$ *forms a Schauder basis in* $L^r(0,1)$ *for all* $r \in (1,\infty)$.

Proof. As in Theorem 6.3,

$$\sum_{j=1}^{\infty} |\eta_{2j+1}(p,q)| < \frac{8\pi_{p,p'}}{\pi^2}\mathcal{C}_{p,p'}\left(\frac{\pi^2}{8}-1\right).$$

According to [12, Proposition 2.5], we have

$$\eta_1(p,p') \geq \frac{8}{\pi\pi_{p,p'}}.$$

Thus because of Lemma 6.6, criterion (6.12) is fulfilled and the result follows. $\square$

Remark 6.1. *The number* $\mathbf{p_1}$ *which appears in Lemma 6.6 as a solution of* (6.13) *approximately equals* $1.487807 < \mathbf{p_0} \approx 1.798658$ *with all digits correct.*

6.4.2 The case $p \in (2,\infty)$ and $q \in (1,\infty)$

Consider the equation

$$\frac{4\pi_{q',p'}}{\pi^3}\left[p'-1+\frac{\pi}{2}(2-p')\right]\left[\left(1-\frac{1}{2^{p'}}\right)\zeta(p')-1\right] = \frac{p-1}{2p-1} - \frac{\pi^2(p-1)}{24(4p-3)}. \tag{6.15}$$

Theorem 6.4. *The sequence* $\mathfrak{E}_{\cos_{p,q}}$ *forms a Schauder basis in* $L^r(0,1)$, $r \in (1,\infty)$ *whenever*

$$\frac{4\pi_{q',p'}}{\pi^3}\left[p'-1+\frac{\pi}{2}(2-p')\right]\left[\left(1-\frac{1}{2^{p'}}\right)\zeta(p')-1\right] < \frac{p-1}{2p-1} - \frac{\pi^2(p-1)}{24(4p-3)}. \tag{6.16}$$

Proof. We know that $\zeta(p') = \frac{1}{1-2^{-p'}}\sum_{j=0}^{\infty}(2j+1)^{-p'}$. In view of (6.9)

$$\begin{aligned}\sum_{j=1}^{\infty} |\eta_{2j+1}(p,q)| &< \frac{4\pi_{q',p'}}{\pi^2}\left[p'-1+\frac{\pi(2-p')}{2}\right]\sum_{j=1}^{\infty}(2j+1)^{-p'} \\ &= \frac{4\pi_{q',p'}}{\pi^2}\left[p'-1+\frac{\pi(2-p')}{2}\right]\left[\left(1-\frac{1}{2^{p'}}\right)\zeta(p')-1\right].\end{aligned}$$

On the other hand, consider $p = p_0^*$ ($p_0^* \approx 1.22$), such that

$$\frac{4p-3}{3p-2} = \frac{\pi^3}{24(\pi-2)}.$$

According to (4.28) from [12],

$$\eta_1(p,q) \geq \frac{\pi(p-1)}{2p-1} - \frac{\pi^3(p-1)}{24(4p-3)}, \quad p \in (p_0^*, \infty). \tag{6.17}$$

In view of the hypothesis, (6.12) holds and the proof is complete. □

6.4.2.1 The case $q = p'$

Inequality (6.16) is equivalent to

$$\tilde{h}(p) := \frac{4\pi_{p,p'}}{\pi^3} \frac{\left[p' - 1 + \frac{\pi}{2}(2-p')\right]\left[\left(1 - \frac{1}{2^{p'}}\right)\zeta(p') - 1\right]}{\frac{p-1}{2p-1} - \frac{\pi^2(p-1)}{24(4p-3)}} < 1$$

given that the denominator of the original fraction is positive for all $p > 1$.

Lemma 6.7. *There exists $p = \mathbf{p_2} \in (2, \frac{14}{5})$ such that*

$$\frac{\tilde{h}(p)}{\pi_{p,p'}} = \frac{1}{4}. \tag{6.18}$$

Moreover, for all $p \in [2, \mathbf{p_2}]$

$$\frac{\tilde{h}(p)}{\pi_{p,p'}} \leq \frac{1}{4}. \tag{6.19}$$

Proof. At $p = 2$ we have $\frac{\tilde{h}(2)}{\pi} = \frac{60}{\pi^3(40-\pi^2)}(\pi^2 - 8) < \frac{1}{4}$. At $p = \frac{14}{5}$,

$$\frac{\tilde{h}(p)}{\pi_{p,p'}} = \frac{4}{9\pi^3} \frac{\left[\frac{5}{9} + \frac{2\pi}{9}\right]\left[\left(1 - \frac{1}{2^{14/9}}\right)\zeta(14/9) - 1\right]}{\frac{1}{23} - \frac{\pi^2}{984}} > \frac{1}{4}.$$

Hence there exists $\mathbf{p_2} \in (2, 14/5)$ such that (6.18) holds.

The function $\frac{p'-1+\frac{\pi}{2}(2-p')}{\frac{p-1}{2p-1}-\frac{\pi^2(p-1)}{24(4p-3)}}$ is positive for any $p\in(2,\infty)$ and

$$\frac{\mathrm{d}}{\mathrm{d}p}\left[\frac{p'-1+\frac{\pi}{2}(2-p')}{\frac{p-1}{2p-1}-\frac{\pi^2(p-1)}{24(4p-3)}}\right]=\frac{1}{p-1}\frac{\frac{(\frac{\pi}{2}-2)p+\frac{\pi}{2}}{(2p-1)^2}+\frac{\pi^2}{24}\frac{(-\frac{3\pi}{2}+4)p-2+\frac{\pi}{2}}{(4p-3)^2}}{\left(\frac{p-1}{2p-1}-\frac{\pi^2(p-1)}{24(4p-3)}\right)^2}$$

is positive whenever $p\in(2,\frac{14}{5})$, which implies that the function is increasing in this interval.

Also notice, Lemma 4.10, that the function $\left(1-\frac{1}{2^{p'}}\right)\zeta(p')-1$ is a positive function increasing in $p\in(2,\infty)$. Therefore the function $\frac{\tilde{h}(p)}{\pi_{p,p'}}$ is increasing in $p\in(2,\frac{14}{5})$. Moreover $\frac{\tilde{h}(p)}{\pi_{p,p'}}<\frac{\tilde{h}(p_2)}{\pi_{p_2,p_2'}}=\frac{1}{4}$ for all $p\in[2,\mathbf{p_2})$ and the result follows. $\square$

Corollary 6.4. *Let $\mathbf{p_2}\in(2,\infty)$ be such that (6.18) holds. Then for any $p\in[2,\mathbf{p_2}]$ the sequence $\mathfrak{E}_{\cos_{p,p'}}$ forms a Schauder basis in $L^r(0,1)$, $r\in(1,\infty)$.*

Proof. Notice that for any $p\in(1,\infty)$ we have $2\le\pi_{p,p'}\le 4$ (Lemma 1.4). According to Lemma 6.7, for all $p\in[2,\mathbf{p_2}]$

$$\tilde{h}(p)\le\frac{16}{\pi^3}\frac{\left[p'-1+\frac{\pi}{2}(2-p')\right]\left[\left(1-\frac{1}{2^{p'}}\right)\zeta(p')-1\right]}{\frac{p-1}{2p-1}-\frac{\pi^2(p-1)}{24(4p-3)}}=4\frac{\tilde{h}(p)}{\pi_{p,p'}}\le 1.$$

Using the same argument as in the proof of Theorem 6.4 for the case $q=p'$ proves that the sequence claimed forms a Schauder basis in $L^r(0,1)$, $r\in(1,\infty)$. $\square$

Remark 6.2. *An approximation of the solution of* (6.18) *shows that* $\mathbf{p_2}\approx 2.526402$ *with all digits correct.*

Corollaries 6.3 and 6.4 identify two refined thresholds $\mathbf{p_1}<\mathbf{p_0}$ and $\mathbf{p_2}>2$ such that for any $p\in[\mathbf{p_1},\mathbf{p_2}]$ the set $\mathfrak{E}_{\cos_{p,p'}}$ generates a Schauder basis for the Banach space $L^r(0,1)$ with $r\in(1,\infty)$. This is an improvement of those available in previous works which were described in Section 2.1.1.

In the notation of Section 4.6, the following results from Section 6.2 will be invoked in several places below. For any $j\in\mathbb{N}$ and $p,q\in(1,\infty)$,

$$|\tau_j(p,q)|\lesssim j^{-\gamma}\qquad\text{and}\qquad|\eta_j(p,q)|\lesssim j^{-\beta},$$

where,

$$\gamma = \begin{cases} 3 & p \in (1,2) \\ p'+1 & p \in (2,\infty) \end{cases} \qquad \text{and} \qquad \beta = \begin{cases} 2 & p \in (1,2) \\ p' & p \in (2,\infty). \end{cases} \tag{6.20}$$

6.5 The regularity of the (p,q)-trigonometric functions

For $s > 0$. Let $H^s(0,1)$ be the (Hilbert) Bessel potential space of order s (Definition 4.1).

Corollary 6.5. *Let $p, q \in (1,\infty)$, let*

$$\sigma_1(p) = \begin{cases} \frac{5}{2} & p \in (1,2), \\ p' + \frac{1}{2} & p \in (2,\infty). \end{cases}$$

Then $\sin_{p,q}(\pi_{p,q}\cdot) \in H^s(0,1)$ for all $s \in [0, \sigma_1(p))$.

Proof. According to (6.20),

$$\sum_{j=1}^{\infty}(1+j^2)^s|\tau_j(p,q)|^2 \lesssim \sum_{j=1}^{\infty} j^{2s-2\gamma} < \infty \qquad \text{if} \qquad s < \gamma - \frac{1}{2}.$$

□

Corollary 6.6. *Let $p, q \in (1,\infty)$, let*

$$\sigma_2(p) = \begin{cases} \frac{3}{2} & p \in (1,2), \\ p' - \frac{1}{2} & p \in (2,\infty). \end{cases}$$

Then $\cos_{p,q}(\pi_{p,q}\cdot) \in H^s(0,1)$ for all $s \in [0, \sigma_2(p))$.

Proof. An analogue of the previous proof implies the regularity of the (p,q)-cosine functions at the specified order s. □

Chapter 7

Shauder Basis in Hardy Spaces

This chapter is motivated by the question whether it is possible to extend the *multi-term* criterion from Chapter 3 to the case of Banach spaces $L^r(0,1)$ for all $r \in (1,\infty)$. That is, using a mechanism analogous to the one developed in the chapter, does there exist an isometry isomorphism U mapping $L^r(0,1)$ onto the Hardy space $H^r(\mathbb{T}^\infty)$ such that $M = U^{-1}\mathscr{M}_m U$, where the operators M and $\mathscr{M}_m$ are mapping $L^r(0,1)$ and $H^r(\mathbb{T}^\infty)$ into themselves, respectively.

For this purpose, we aim to develop an approach in general which characterises bases properties of systems in Hardy spaces $H^r(\mathbb{T}^\infty)$ in terms of the multipliers. This study together with the research developed in Chapter 3 will hopefully facilitate the translation of the same study to the Banach space setting $L^r(0,1)$ for $r \in (1,\infty)$.

Given φ any well-defined function on $\mathbb{T}^\infty$. For $r \in (1,\infty)$, let $\{\mathfrak{h}_n\}_{n\in\mathbb{N}}$ be a basis in $H^r(\mathbb{T}^\infty)$. We wish to investigate the conditions under which the family $\mathfrak{H}_\varphi = \{\varphi\mathfrak{h}_n\}_{n\in\mathbb{N}}$ forms a basis of $H^r(\mathbb{T}^\infty)$.

In Section 7.1 we shall characterise the properties of multiplication operators $\mathcal{T}_\varphi$ defined on Hardy spaces $H^r(G)$ when looking at $G = \mathbb{D}^n, \mathbb{T}^n$ and $\mathbb{T}^\infty$. Then in Section 7.2 we illustrate some general criteria that allow us to investigate the bases properties of $\mathfrak{H}_\varphi$ in $H^r(\mathbb{T}^\infty)$ for $r \in (1,\infty)$.

7.1 Multiplication operators on Hardy spaces

For $r \in (1, \infty)$. Let $G = \mathbb{D}^n, \mathbb{T}^n$ or $\mathbb{T}^\infty$ and let $H^r(G)$ be the Hardy space on G. A function φ analytic on G is by definition called a *symbol*. A measurable function $\varphi : G \longrightarrow \mathbb{C}$ is a multiplier on $H^r(G)$, if for any $F \in H^r(G)$ we have $\varphi F \in H^r(G)$. We denote the collection of all multipliers on $H^r(G)$ by $\mathcal{S}(H^r(G))$, i.e.,

$$\mathcal{S}(H^r(G)) = \{\varphi : G \longrightarrow \mathbb{C} : \varphi F \in H^r(G) \text{ for all } F \in H^r(G)\}.$$

Let $\mathcal{T}_\varphi : H^r(G) \longrightarrow H^r(G)$ be the multiplication operator on $H^r(G)$ defined by,

$$\mathcal{T}_\varphi F(z) = \varphi(z)F(z), \qquad \forall F \in H^r(G). \tag{7.1}$$

This section will study multipliers algebra on Hardy spaces $H^r(G)$ for $G = \mathbb{D}^n, \mathbb{T}^n$ and $\mathbb{T}^\infty$. It will also show the calculations for the operator norm of $\mathcal{T}_\varphi$ in each case.

7.1.1 Multipliers algebra on Hardy spaces $H^r(\mathbb{D}^n)$ and $H^r(\mathbb{T}^n)$

Theorem 7.1. *The function φ is in $H^\infty(\mathbb{D}^n)$ if, and only if, $\varphi \in \mathcal{S}(H^r(\mathbb{D}^n))$. Moreover,*

$$\|\mathcal{T}_\varphi\|_{\mathcal{B}(H^r(\mathbb{D}^n))} = \|\varphi\|_\infty.$$

Proof. Suppose that $\varphi \in H^\infty(\mathbb{D}^n)$. For all $F \in H^r(\mathbb{D}^n)$,

$$\begin{aligned}
\|\mathcal{T}_\varphi F\|_{H^r(\mathbb{D}^n)} &= \left(\sup_{0\le\rho<1} \int_{\mathbb{T}^n} |\varphi_\rho(z)F_\rho(z)|^r \mathrm{d}\sigma_n(z)\right)^{1/r} \\
&\le \sup_{z\in\mathbb{D}^n} |\varphi(z)| \left(\sup_{0\le\rho<1} \int_{\mathbb{T}^n} |F_\rho(z)|^r \mathrm{d}\sigma_n(z)\right)^{1/r} \\
&= \|\varphi\|_\infty \|F\|_{H^r(\mathbb{D}^n)} < \infty.
\end{aligned} \tag{7.2}$$

We conclude that φ is a multiplier on $H^r(\mathbb{D}^n)$.

Conversely, if $\varphi \in \mathcal{S}(H^r(\mathbb{D}^n))$, then $\mathcal{T}_\varphi \in \mathcal{B}(H^r(\mathbb{D}^n))$ as a consequence of the Closed Graph Theorem [23, Chapter III, Section 5, pp. 166]. Since $1 \in H^r(\mathbb{D}^n)$, then $\varphi = \varphi.1 \in H^r(\mathbb{D}^n)$. Consider h_ξ from (2.11) with norm $\|h_\xi\|_{H^r(\mathbb{D}^n)} = 1$. Then

$\varphi h_\xi \in H^r(\mathbb{D}^n)$ and the inequality (2.10) holds for any $\xi \in \mathbb{D}^n$. That is

$$\begin{aligned} |\varphi(\xi)|^r |h_\xi(\xi)|^r = |\varphi(\xi) h_\xi(\xi)|^r &\leq \prod_{j=1}^{n} \frac{1}{1-|\xi_j|^2} \, \|\varphi h_\xi\|^r_{H^r(\mathbb{D}^n)} \\ &\leq \prod_{j=1}^{n} \frac{1}{1-|\xi_j|^2} \, \|\mathcal{T}_\varphi\|^r_{\mathcal{B}(H^r(\mathbb{D}^n))} \, \|h_\xi\|^r_{H^r(\mathbb{D}^n)}. \end{aligned}$$

Since (2.10) is sharp for h_ξ, then

$$|\varphi(\xi)|^r \leq \|\mathcal{T}_\varphi\|^r_{\mathcal{B}(H^r(\mathbb{D}^n))}.$$

Taking supremum over $\mathbb{D}^n$ of both sides yields

$$\|\varphi\|_\infty \leq \|\mathcal{T}_\varphi\|_{\mathcal{B}(H^r(\mathbb{D}^n))} < \infty. \tag{7.3}$$

Therefore, $\varphi \in H^\infty(\mathbb{D}^n)$. From (7.2) and (7.3) the identity follows. $\square$

Theorem 7.2. *If $\varphi \in H^\infty(\mathbb{D}^n)$ and $\varphi(z) \neq 0$ on the closed polydisc $\overline{\mathbb{D}}^n$, then $\mathcal{T}_\varphi^{-1} : H^r(\mathbb{D}^n) \longrightarrow H^r(\mathbb{D}^n)$ exists such that*

$$\mathcal{T}_\varphi^{-1} F(z) = \varphi^{-1}(z) F(z), \qquad \forall F \in H^r(\mathbb{D}^n).$$

Moreover, $\|\mathcal{T}_\varphi^{-1}\|^{-1}_{\mathcal{B}(H^r(\mathbb{D}^n))} = \inf_{z \in \mathbb{D}^n} |\varphi(z)|$.

Proof. Observe that $\varphi(z)\varphi^{-1}(z) = \varphi^{-1}(z)\varphi(z) = 1$ for all $z \in \overline{\mathbb{D}}^n$. This implies that $\mathcal{T}_\varphi$ is one to one and onto. Hence $\mathcal{T}_\varphi^{-1}$ exists. For any $F \in H^r(\mathbb{D}^n)$,

$$\mathcal{T}_\varphi \left(\mathcal{T}_\varphi^{-1} F(z) \right) = \mathcal{T}_\varphi \left(\varphi^{-1}(z) F(z) \right) = \varphi(z)\varphi^{-1}(z) F(z) = F(z)$$

and

$$\mathcal{T}_\varphi^{-1} \left(\mathcal{T}_\varphi F(z) \right) = \mathcal{T}_\varphi^{-1} \left(\varphi(z) F(z) \right) = \varphi^{-1}(z)\varphi(z) F(z) = F(z).$$

Consequently $\mathcal{T}_\varphi^{-1} = \mathcal{T}_{\varphi^{-1}}$. According to Theorem 7.1, $\mathcal{T}_\varphi \in \mathcal{B}(H^r(\mathbb{D}^n))$ which by the Inverse Mapping Theorem implies that $\mathcal{T}_\varphi^{-1} \in \mathcal{B}(H^r(\mathbb{D}^n))$. Moreover, in view of

Theorem 7.1, we have

$$\|\mathcal{T}_\varphi^{-1}\|_{\mathcal{B}(H^r(\mathbb{D}^n))} = \|\varphi^{-1}\|_\infty = \sup_{z\in\mathbb{D}^n} |\varphi^{-1}(z)| = \left(\inf_{z\in\mathbb{D}^n} |\varphi(z)|\right)^{-1},$$

and the result follows. □

Note 7.1. *According to the Minimum Modulus Principle [26, Chapter I], the infimum lies on the distinguished boundary $\mathbb{T}^n$. The results from this study also hold when $\mathcal{T}_\varphi$ is the multiplication operator on the Hardy space $H^r(\mathbb{T}^n)$. This observation is due to Remark 2.1.*

7.1.2 Multipliers algebra on Hardy spaces $H^r(\mathbb{T}^\infty)$

Theorem 7.3. *The function φ is in $H^\infty(\mathbb{T}^\infty)$ if, and only if, $\varphi \in \mathcal{S}(H^r(\mathbb{T}^\infty))$. Moreover,*

$$\|\mathcal{T}_\varphi\|_{\mathcal{B}(H^r(\mathbb{T}^\infty))} = \|\varphi\|_\infty.$$

Proof. Assume that $\varphi \in H^\infty(\mathbb{T}^\infty)$. For any $F \in H^r(\mathbb{T}^\infty)$,

$$\begin{aligned}\|\mathcal{T}_\varphi F\|_{H^r(\mathbb{T}^\infty)} &= \left(\int_{\mathbb{T}^\infty} |\varphi(z)F(z)|^r \,\mathrm{d}\sigma(z)\right)^{1/r} \\ &\leq \sup_{z\in\mathbb{T}^\infty} |\varphi(z)| \left(\int_{\mathbb{T}^\infty} |F(z)|^r \mathrm{d}\sigma(z)\right)^{1/r} \\ &= \|\varphi\|_\infty \|F\|_{H^r(\mathbb{T}^\infty)}, \end{aligned} \tag{7.4}$$

which implies that $\varphi F \in H^r(\mathbb{T}^\infty)$. Hence, $\varphi \in \mathcal{S}(H^r(\mathbb{T}^\infty))$.

Now assume that $\varphi \in \mathcal{S}(H^r(\mathbb{T}^\infty))$. By the Closed Graph Theorem [23, Chapter III, Section 5, pp. 166], the operator $\mathcal{T}_\varphi$ is bounded. Since $1 \in H^r(\mathbb{T}^\infty)$, $\varphi \in H^r(\mathbb{T}^\infty)$. Moreover, $\mathcal{T}_\varphi F = \varphi F$ for all $F \in H^r(\mathbb{T}^\infty)$. Now, since $\|\varphi F\|_{H^r(\mathbb{T}^\infty)} \leq \|\mathcal{T}_\varphi\|_{\mathcal{B}(H^r(\mathbb{T}^\infty))}\|F\|_{H^r(\mathbb{T}^\infty)}$, we get, by successively plugging in $F = 1, \varphi, \ldots, \varphi^{j-1}$ (for $j \in \mathbb{N}$ fixed), that $\|\mathcal{T}_\varphi\|_{\mathcal{B}(H^r(\mathbb{T}^\infty))} \geq \|\varphi^j\|_{H^r(\mathbb{T}^\infty)}^{1/j}$. That is, for all $z \in \mathbb{T}^\infty$, If $F(z) = 1$ then

$$\|\varphi\|_{H^r(\mathbb{T}^\infty)} \leq \|\mathcal{T}_\varphi\|_{\mathcal{B}(H^r(\mathbb{T}^\infty))}.$$

If $F(z) = \varphi(z)$, then

$$\|\varphi^2\|_{H^r(\mathbb{T}^\infty)} \leq \|\mathcal{T}_\varphi\|_{\mathcal{B}(H^r(\mathbb{T}^\infty))}\|\varphi\|_{H^r(\mathbb{T}^\infty)} \leq \|\mathcal{T}_\varphi\|^2_{\mathcal{B}(H^r(\mathbb{T}^\infty))}.$$

Hence, after j iterations we get that

$$\|\mathcal{T}_\varphi\|_{\mathcal{B}(H^r(\mathbb{T}^\infty))} \geq \|\varphi^j\|^{1/j}_{H^r(\mathbb{T}^\infty)}.$$

Let $\delta \in (0, \|\varphi\|_\infty]$. Set

$$S_\delta = \{z \in \mathbb{T}^\infty : |\varphi(z)| \geq \|\varphi\|_\infty - \delta\},$$

which is a subspace of $\mathbb{T}^\infty$ with a finite positive measure $\sigma(S_\delta) \in (0, 1]$. Then,

$$\begin{aligned}\|\mathcal{T}_\varphi\|_{\mathcal{B}(H^r(\mathbb{T}^\infty))} \geq \|\varphi^j\|^{1/j}_{H^r(\mathbb{T}^\infty)} &= \left(\int_{\mathbb{T}^\infty} |\varphi(z)|^{rj} \mathrm{d}\sigma(z)\right)^{\frac{1}{rj}} \\ &\geq (\|\varphi\|_\infty - \delta)\,(\sigma(S_\delta))^{\frac{1}{rj}}.\end{aligned}$$

When $j \to \infty$, $\|\mathcal{T}_\varphi\|_{\mathcal{B}(H^r(\mathbb{T}^\infty))} \geq \|\varphi\|_\infty - \delta$. Hence,

$$\operatorname*{ess\,inf}_{\delta} \|\mathcal{T}_\varphi\|_{\mathcal{B}(H^r(\mathbb{T}^\infty))} = \|\varphi\|_\infty, \tag{7.5}$$

and $\varphi \in H^\infty(\mathbb{T}^\infty)$. From (7.4) and (7.5) the result

$$\|\mathcal{T}_\varphi\|_{\mathcal{B}(H^r(\mathbb{T}^\infty))} = \|\varphi\|_\infty$$

follows. □

Theorem 7.4. *If $\varphi \in H^\infty(\mathbb{T}^\infty)$ and $\varphi(z) \neq 0$ on the closed polydisc $\overline{\mathbb{D}}^\infty$, then $\mathcal{T}_\varphi^{-1} : H^r(\mathbb{T}^\infty) \longrightarrow H^r(\mathbb{T}^\infty)$ exists such that*

$$\mathcal{T}_\varphi^{-1} F(z) = \varphi^{-1}(z) F(z), \qquad \forall F \in H^r(\mathbb{T}^\infty).$$

Moreover, $\|\mathcal{T}_\varphi^{-1}\|^{-1}_{\mathcal{B}(H^r(\mathbb{T}^\infty))} = \inf_{z \in \mathbb{T}^\infty} |\varphi(z)|$.

Proof. The proof is similar to that of Theorem 7.2 for Hardy spaces $H^r(\mathbb{T}^\infty)$, hence it is omitted. $\square$

Remark 7.1. *Any $\varphi \in H^r(\mathbb{T}^\infty)$ has a power series representation in several complex variables. This series is indexed at $\nu \in \mathbb{N}_0^\infty$ and therefore it depends only on a finite number of variables z_j. Hence we may restrict our attention to $H^r(\mathbb{T}^n)$ (or to $H^r(\mathbb{D}^n)$ by analytic continuity) for some $n \in \mathbb{N}$ finite.*

7.2 Bases properties of $\mathfrak{H}_\varphi$ in $H^r(\mathbb{T}^\infty)$

Following the approach of Section 3.2. The family of dilations of the sequence of monomials, $\mathfrak{H}_\varphi$, is a basis of $H^r(\mathbb{T}^\infty)$ if, and only if, the operator $\mathcal{T}_\varphi$ is a homeomorphism (see Theorem 2.2). We arrive at a principle which examines bases properties of such sequences in terms of the analytic properties of φ and the localisation of its zeros with respect to $\overline{\mathbb{D}}^\infty$.

Theorem 7.5. *If $\varphi \in H^\infty(\mathbb{T}^\infty)$ and $\varphi(z) \neq 0$ on $\overline{\mathbb{D}}^\infty$, then $\mathfrak{H}_\varphi$ is a basis of $H^r(\mathbb{T}^\infty)$ for all $r \in (1, \infty)$.*

Proof. According to Theorems 7.3 and 7.4, the operator $\mathcal{T}_\varphi$ is a homeomorphism. Since $\{\mathfrak{h}_n\}_{n\in\mathbb{N}}$ is a basis of $H^r(\mathbb{T}^\infty)$ for $r \in (1, \infty)$ (Theorem 2.11), so is $\mathfrak{H}_\varphi$. The result follows. $\square$

Let $\{\mathfrak{h}_n\}_{n\in\mathbb{N}}$ be the basis of monomials (2.13) in $H^r(\mathbb{T}^\infty)$. A function $\varphi \in H^r(\mathbb{T}^\infty)$ has a Fourier expansion in several complex variables

$$\varphi(z) = \sum_{n\in\mathbb{N}} \widehat{\varphi}_n \ \mathfrak{h}_n(z) \iff \varphi(z) = \sum_{\nu\in\mathbb{N}_0^\infty} \widehat{\varphi}_\nu \ \mathfrak{h}_n(z), \quad \forall z \in \mathbb{T}^\infty. \tag{7.6}$$

The Fourier coefficients $\widehat{\varphi}_\nu$ are identified to correspond to $\widehat{\varphi}_n$ for all $\nu = (\nu_j)_{j\in\mathbb{N}} \in \mathbb{N}_0^\infty$ and $n \in \mathbb{N}$ such that

$$\widehat{\varphi}_n = \int_{\mathbb{T}^\infty} \varphi(z)\overline{h_n(z)}\mathrm{d}\sigma(z),$$

where $\bar{z}$ denotes the complex conjugate of z (see, Definition 2.7 for more details).

Consider (7.6) such that $\sum_{n=1}^{\infty} |\widehat{\varphi}_n| < \infty$. Let $\mathcal{F} \subset \mathbb{N}$ be finite such that $1 \in \mathcal{F}$. Decompose $\varphi(z)$ into

$$\varphi(z) = m(z) + v(z),$$

such that

$$m(z) = \sum_{n \in \mathcal{F}} \widehat{\varphi}_n \; z_1^{\nu_{\mathfrak{p}_1}(n)} \cdots z_d^{\nu_{\mathfrak{p}_d}(n)} \tag{7.7}$$

and

$$v(z) = \sum_{n \in \mathbb{N} \setminus \mathcal{F}} \widehat{\varphi}_n \; z_1^{\nu_{\mathfrak{p}_1}(n)} \cdots z_k^{\nu_{\mathfrak{p}_k}(n)},$$

where $\nu_{\mathfrak{p}}(n) = 0$ whenever $\mathfrak{p}$ is not a divisor of n. For simplicity we will sometimes write ν_j instead of $\nu_{\mathfrak{p}_j}$.

Adopting the argument used in Section 3.2 in the context of Hardy spaces $H^r(\mathbb{T}^\infty)$, $r \in (1, \infty)$ yields the following:

If

$$\left.\begin{array}{l} m(z) \neq 0, \;\; z \in \overline{\mathbb{D}}^d \\ \|v\|_\infty < \|m^{-1}\|_\infty^{-1} \end{array}\right\} \quad \Rightarrow \quad \mathfrak{H}_\varphi \text{ is a Schauder basis in } H^r(\mathbb{T}^\infty). \tag{7.8}$$

Note that the left hand side of the implication in (7.8) proves the invertibility of $\varphi(z) = m(z)\,[1 + m^{-1}(z) v(z)]$ in the closed polydisc $\overline{\mathbb{D}}^\infty$.

Properties of the functions m and $1/m$

Consider m from (7.7) with $\widehat{\varphi}_1 \neq 0$. Suppose that m is invertible in $\overline{\mathbb{D}}^d$. Then

$$1/m(z) = \frac{1}{\widehat{\varphi}_1} \left(b_1 + \sum_{n=2}^{\infty} b_n z_1^{\mu_{\mathfrak{p}_1}(n)} \ldots z_d^{\mu_{\mathfrak{p}_d}(n)} \right), \tag{7.9}$$

the coefficients b_n are given by, $b_1 = 1$ and

$$b_n = \sum_{\mathfrak{r}_1^{\alpha_1} \ldots \mathfrak{r}_d^{\alpha_d} = n} (-1)^{\sum_{j=1}^d \alpha_j} \, \delta_n \left(\frac{\widehat{\varphi}_{\mathfrak{r}_1}}{\widehat{\varphi}_1} \right)^{\alpha_1} \ldots \left(\frac{\widehat{\varphi}_{\mathfrak{r}_d}}{\widehat{\varphi}_1} \right)^{\alpha_d}, \; n > 1$$

such that

$$\delta_n = \frac{\left(\sum_{j=1}^{d} \alpha_j\right)!}{\alpha_1! \dots \alpha_d!},$$

[20, Section 5]. The summation in $b_n, n > 1$ runs over all the positive integers $\mathfrak{r}_j > 1$ (including the prime numbers) dividing n, with multiplicities $\alpha_j = 0$ when n is not a multiple of $\mathfrak{r}_j$. If we multiply $m(z)$ by $1/m(z)$,

$$b_1 \widehat{\varphi}_1 = \widehat{\varphi}_1 \quad \text{and} \quad \sum_{\mathfrak{q} \in \mathbb{N}: \mathfrak{q} | n} b_{\mathfrak{q}} \widehat{\varphi}_{n/\mathfrak{q}} = 0 \tag{7.10}$$

The existence of $1/m(z)$ is guaranteed, if

$$\widehat{\varphi}_1 \neq 0 \qquad \text{and} \qquad \sum_{j=2}^{\infty} |\widehat{\varphi}_n| < |\widehat{\varphi}_1|.$$

But it is not the only way that invertibility is ensured (see for example, Theorem 7.7). Moreover, $1/m(z)$ has an infinite series expansion in exactly the same variables $z_1, \dots, z_d$ as in $m(z)$ with possible different multiplicities $\mu_{\mathfrak{p}}(n)$ such that $\mu_{\mathfrak{p}}(n) = 0$ whenever $\mathfrak{p}$ is not dividing n. On the other hand, $1/m(z)$ can also be considered as a function in infinitely many variables $z_j,\ j \in \mathbb{N}$ with corresponding multiplicities $\mu_{\mathfrak{p}_j}(n)$ such that $\mu = (\mu_{\mathfrak{p}_j}(n))_{j \in \mathbb{N}} \in \mathbb{N}_0^{\infty}$; the series in $1/m(z)$ converges pointwise for all $z \in \overline{\mathbb{D}}^d$ and also in the L^r-norm (see statement (7.10), Expansion (Taylor) Theorem [24, Theorem 1, pp. 79] and M. Riesz Lemma).

The following two examples illustrate the structure of $1/m$.

Examples 7.1. *For $\widehat{\varphi}_1 \neq 0$ and $|\widehat{\varphi}_2| < |\widehat{\varphi}_1|$, define*

$$m(z) = \widehat{\varphi}_1 + \widehat{\varphi}_2 z_1, \qquad z_1 \in \mathbb{D}.$$

Then,

$$b_1 = 1 \qquad \text{and} \qquad b_n = \begin{cases} (-1)^k \left(\frac{\widehat{\varphi}_2}{\widehat{\varphi}_1}\right)^k & n = 2^k, k \in \mathbb{N} \\ 0 & \text{otherwise,} \end{cases}$$

and $1/m(z)$ *has the series expansion*

$$1/m(z) = \frac{1}{\widehat{\varphi}_1} \sum_{k=0}^{\infty} (-1)^k \left(\frac{\widehat{\varphi}_2}{\widehat{\varphi}_1} \right)^k z_1^k, \qquad \forall z_1 \in \mathbb{D}$$

Examples 7.2. *For* $\widehat{\varphi}_1 \neq 0$ *and* $|\widehat{\varphi}_2| + |\widehat{\varphi}_3| < |\widehat{\varphi}_1|$, *define*

$$m(z) = \widehat{\varphi}_1 + \widehat{\varphi}_2 z + \widehat{\varphi}_3 \xi, \qquad (z, \xi) \in \mathbb{D}^2.$$

Then,

$$b_n = \begin{cases} (-1)^{j+k} \frac{(j+k)!}{j!\,k!} \left(\frac{\widehat{\varphi}_2}{\widehat{\varphi}_1} \right)^j \left(\frac{\widehat{\varphi}_3}{\widehat{\varphi}_1} \right)^k & n = 2^j 3^k, j, k \in \mathbb{N}_0 \\ 0 & \textit{otherwise} \end{cases}$$

and $1/m(z)$ *takes the form*

$$1/m(z) = \frac{1}{\widehat{\varphi}_1} \sum_{j,k=0}^{\infty} (-1)^{j+k} \frac{(j+k)!}{j!\,k!} \left(\frac{\widehat{\varphi}_2}{\widehat{\varphi}_1} \right)^j \left(\frac{\widehat{\varphi}_3}{\widehat{\varphi}_1} \right)^k z^j\, \xi^k, \qquad \forall (z, \xi) \in \mathbb{D}^2.$$

Theorems 7.6, 7.7 and Corollary 7.1 correspond to those of Section 3.2 in the context of Hardy spaces $H^r(\mathbb{T}^\infty)$. Note that a *multi-term* criterion for $H^r(\mathbb{T}^\infty)$ will be valid for all $r \in (1, \infty)$ and not only $r = 2$. Some of the proofs are similar and are therefore omitted.

When $m(z) = \widehat{\varphi}_1$ for all $z \in \mathbb{D}$ such that $\widehat{\varphi}_1 \neq 0$. Then

Theorem 7.6. *If*

$$\sum_{n=2}^{\infty} |\widehat{\varphi}_n| < |\widehat{\varphi}_1|,$$

the family $\mathfrak{H}_\varphi$ *forms a Schauder basis of* $H^r(\mathbb{T}^\infty)$ *for* $r \in (1, \infty)$.

Now consider the case $\mathcal{F} = \{1, \mathfrak{p}, \mathfrak{p}^2\}$. Then

$$m(z) = \widehat{\varphi}_1 + \widehat{\varphi}_{\mathfrak{p}} z + \widehat{\varphi}_{\mathfrak{p}^2} z^2, \qquad z \in \mathbb{D}.$$

Theorem 7.7. *Assume that*

$$0 < \widehat{\varphi}_{\mathfrak{p}^2} < \widehat{\varphi}_1 \quad \text{and} \quad \widehat{\varphi}_{\mathfrak{p}^2} + \widehat{\varphi}_1 > |\widehat{\varphi}_{\mathfrak{p}}|. \tag{7.11}$$

Either of the following two conditions ensure that $\mathfrak{H}_\varphi$ is a Schauder basis of $H^r(\mathbb{T}^\infty)$.

(a) $|\widehat{\varphi}_{\mathfrak{p}}(\widehat{\varphi}_{\mathfrak{p}^2} + \widehat{\varphi}_1)| \geq |4\widehat{\varphi}_{\mathfrak{p}^2}\widehat{\varphi}_1|$ *and*

$$\sum_{j\in\mathbb{N}\setminus\{1,\mathfrak{p}^2\}} |\widehat{\varphi}_j| < \widehat{\varphi}_1 + \widehat{\varphi}_{\mathfrak{p}^2}, \tag{7.12}$$

(b) $|\widehat{\varphi}_{\mathfrak{p}}(\widehat{\varphi}_{\mathfrak{p}^2} + \widehat{\varphi}_1)| < |4\widehat{\varphi}_{\mathfrak{p}^2}\widehat{\varphi}_1|$ *and*

$$\sum_{j\in\mathbb{N}\setminus\{1,\mathfrak{p},\mathfrak{p}^2\}} |\widehat{\varphi}_j| < (\widehat{\varphi}_1 - \widehat{\varphi}_{\mathfrak{p}^2})\sqrt{1 - \frac{\widehat{\varphi}_{\mathfrak{p}}^2}{4\widehat{\varphi}_1\widehat{\varphi}_{\mathfrak{p}^2}}}. \tag{7.13}$$

Proof. Due to (7.11), $m(z)$ has no zeros in the closed unit disc $\overline{\mathbb{D}}$. □

Corollary 7.1 combines the two cases, $\mathcal{F} = \{1\}$ and $\mathcal{F} \neq \{1\}$, using an approach amenable to computability either analytical or by accurate numerical means of all the quantities involved.

Assume that

$$|\widehat{\varphi}_n| \leq \phi_n, \qquad \forall n \in \mathbb{N}$$

for a sequence $\{\phi_n\}_{n=1}^\infty \in \ell^1(\mathbb{N})$. Set $\Phi = \sum_{n=1}^\infty \phi_n < \infty$. Set

$$d = \begin{cases} 1 & \mathcal{F} = \{1\} \\ \#\mathbb{P}(\mathcal{F}) & \mathcal{F} \neq \{1\}. \end{cases}$$

Corollary 7.1. *Let $k, d \in \mathbb{N}$ be fixed. Let*

$$\mathcal{F} = \{1,\, \mathfrak{p}_1,\, \mathfrak{p}_1^2,\, \dots,\, \mathfrak{p}_d,\, \mathfrak{p}_d^2\}.$$

Let

$$m(z) = \widehat{\varphi}_1 + \sum_{j=1}^d \left(\widehat{\varphi}_{\mathfrak{p}_j} z_j + \widehat{\varphi}_{\mathfrak{p}_j^2} z_j^2\right), \qquad \forall z \in \mathbb{D}^d$$

and

$$\omega = \inf\{|m(z)| : z \in \mathbb{T}^d\}.$$

If

$$\sum_{j\in\mathcal{F}\setminus\{1\}} |\widehat{\varphi}_j| < \widehat{\varphi}_1 \tag{7.14}$$

and

$$\omega - \Phi + \sum_{j\in\mathcal{F}} |\widehat{\varphi}_j| + \sum_{j=1}^{k} (\phi_j - |\widehat{\varphi}_j|) > 0, \tag{7.15}$$

then $\mathfrak{H}_\varphi$ is a Schauder basis of $H^r(\mathbb{T}^\infty)$.

Proof. The proof is the same as Corollary 3.2. $\square$

We end this chapter with the following open questions (see Chapter 8 for a brief description) which could be somewhat hard, taking into account the intractable nature of the spaces $H^r(\mathbb{D}^\infty)$ for $r \in (1, \infty]$, e.g., Note 2.1.

Question (1): Does the set of multipliers on $H^r(\mathbb{D}^\infty)$ coincide with $H^\infty(\mathbb{D}^\infty)$?

Question (2): Is it possible to extend the *multi-term* criterion in Lebesgue spaces (Chapter 3) to the Banach space setting?

Chapter 8

Future Work

8.1 Fatou's Theorem

The research objective is to conduct a deeper study of the boundary behaviour of functions in the Hardy spaces $H^r(\mathbb{T}^\infty)$, $(r > 1)$ for the infinite dimensional polydisc, using tools and techniques developed in interrelated areas of analysis, with a goal of settling open and important questions. In recent years, there has been a renewed interest in these spaces, mainly due to their connection to Dirichlet series and thereby to analytic number theory.

The aim of the research is to show to which extent the standard Fatou-type result remain valid for the infinite dimensional case. The famous scalar case of Fatou's Theorem was at the origin of a large body of research in the 20th-century mathematics under the name of bounded analytic functions which states: An analytic function bounded in the L^r-norm of the unit disc has (radial) non-tangential limits almost everywhere on the unit circle and this function is the Poisson integral of the limit. This was then shown to remain true for the finite dimensional case with almost no restrictions to the radial and even the non-tangential convergence [34]. First steps in this direction were set in [36] for spaces $H^\infty(\mathbb{T}^\infty)$. Further investigation was then settled in [2], where the authors illustrate an analysis of the boundary radial approach of a specific type for which the standard Fatou-type results remain valid. The analysis is mainly based on the work of Helson who introduced the so-called vertical limit functions into the theory of Dirichlet series which allows the analytical extension of Dirichlet series up to the imaginary axis, with a concise consideration of results on bounded point evaluations developed by B. J. Cole and T. W. Gamelin [10].

The problem can be phrased as a question about certain aspects of the theory of holomorphic functions of several complex variables which have not yet attracted much attention although they are very closely related to classical analysis. Briefly, the object is to see how much of our extremely detailed knowledge about holomorphic functions in the unit disc and even the finite dimensional polydisc, such as boundary values, distribution of zeros as related to growth restrictions, factorisation theorem, invariant subspaces, interpolation theorems can be carried over to an analogous situation in several complex variables, namely to infinite dimensional polydiscs.

The theory of the Hardy spaces $H^r(\mathbb{D}^\infty)$ is a mixture of real and complex analysis and this research will introduce this theory with a special emphasis on the results and techniques needed for the main study. The theory is based on the analysis of the non-tangential (Hardy-Littlewood) maximal functions of a function on the polycircle $\mathbb{T}^\infty$ as an auxiliary tool to measure the behaviour of the Poisson integral; the subharmonicity of $|f|^r$ and $\log|f|$ for an analytic function f on the polydisc.

This research will utilise knowledge and techniques from these broad areas of analysis to provide an array of tools with which to approach the challenging questions raised in this problem. As a consequence, we will address questions that relate the boundary value problem to investigations concerning the embedding problem for Dirichlet-Hardy spaces $\mathscr{H}^r$ (introduced in Section 2.2.5) when $r > 0$, which states whether the analogue of the Carlson Embedding problem holds for $r \neq 2k$, $k \in \mathbb{N}$, a question first considered by F. Bayart in [4] and was then addressed in [36] showing that it is true when r is an even integer.

8.2 Multipliers Algebra of $\mathscr{H}^r$ spaces

The objective of this research is to conduct an in depth study of the multipliers algebra of $\mathscr{H}^r$ spaces for all $r \in (1, \infty)$ and investigate some of their applications regarding the basis properties of sets of dilated functions in the Banach space

$L^r(0,1)$.

Hardy spaces of Dirichlet series $\mathscr{H}^2$, defined as the set of Dirichlet series with square summable coefficients, were first studied in a paper by H. Hedenmalm, P. Lindqvist and K. Seip [20], where the authors introduce a framework that measures Riesz basis and completeness properties of a family of dilated functions in $L^2(0,1)$ in terms of the elements of $\mathscr{H}^2$. Among other results they characterise the set of multipliers of the space $\mathscr{H}^2$ and state that, [20, Theorem 3.1], it is precisely the set of bounded analytic functions in the right half-plane $\mathbb{C}^+$ which can be represented as a convergent Dirichlet series. Moreover, it was shown that the operator norm of a multiplier is its supremum norm in $\mathbb{C}^+$. In 2002, F. Bayart [4] introduced Hardy-Dirichlet spaces $\mathscr{H}^r$ for $r \in [1,\infty)$ based on Bohr's vision of Dirichlet series and ergodic theorems. The former dates back to 1913 and suggests the association of a Dirichlet series with a Fourier series on the infinite dimensional polydisc. These tools allow the identification of $\mathscr{H}^r$ spaces with the Hardy spaces of infinite dimensional polydisc $H^r(\mathbb{D}^\infty)$ when $r \in [1,\infty)$ as the closure of the set of Dirichlet polynomials $p(s) = \sum_{j=1}^N a_j j^{-s}$ with $s = \sigma + it, \sigma > 0$ and $t \in \mathbb{R}$, in terms of the norm

$$\|p(s)\|_r^r = \lim_{T\to\infty} \int_{-T}^{T} |p(it)|^r \mathrm{d}t.$$

In addition, the author introduced the set of multipliers of the spaces $\mathscr{H}^r$ for $r \in [1,\infty)$ [4, Theorem 7], showing that it is equivalent to the set of analytic functions in $\mathbb{C}^+$ which can be represented by a convergent Dirichlet series in some half plane.

The research proposed will involve an analysis of the multipliers of Hardy-Dirichlet spaces by finding a precise expression for their operator norm accompanied by an understanding of the concept of cyclic functions of these spaces which are modelled as the ones of the Hardy spaces of the infinite dimensional polydisc. This will hopefully aid the formulation of a framework extending that of the seminar work of Beurling dating back to 1945 which enables the association of the Fourier series of a continuous periodic function in $L^r(0,1), r \in (1,\infty)$ with a corresponding Dirichlet series. The framework will be analogous to that of Riesz basis of the spaces $\mathscr{H}^2$ [20] aiming at studying Schauder basis and/or completeness properties of a family of dilated functions in $L^r(0,1)$ in terms of the analytic properties of the Dirichlet series. The outcome of this research will provide an alternative approach to that established in Chapter 3 of this thesis which is mainly based on the analyticity of the corresponding Dirichlet series.

Appendix A

Appendix

A.1 Properties of $I_p(x)$

We begin by recalling the following property established by Lemma 1.7*(a)* and statement (1.15). For any $p, q \in (1, \infty)$ such that $p < q$,

$$1 < \frac{I_q(x)}{I_p(x)} < \frac{\pi_p}{\pi_q}, \qquad x \in (0, 1]. \tag{A.1}$$

Lemma A.1. *For $p > 1$ fixed. The function $I_p(x)$ is monotonically increasing and convex in $x \in [0, 1]$.*

Proof. Since

$$\frac{\mathrm{d}}{\mathrm{d}x} I_p(x) = \frac{2}{\pi_p}(1 - x^p)^{-\frac{1}{p}} > 0$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} I_p(x) = \frac{2}{\pi_p} x^{p-1}(1 - x^p)^{-\frac{1}{p}-1} \geq 0.$$

The result follows. $\square$

Lemma A.2. *Let $k \in \mathbb{N}$ be fixed. Let $0 \leq y < x \leq 1$ be such that $\cos\left(\frac{k\pi}{2} I_p(u)\right)$ is decreasing for all $u \in [y, x]$. Then,*

$$\int_y^x \cos\left(\frac{k\pi}{2} I_p(u)\right) \mathrm{d}u > \mathcal{I}_{k,p}(y, x)$$

where

$$\mathcal{I}_{k,p}(y,x) := \frac{2}{k\pi}(x-y)\frac{\sin\left(\frac{k\pi}{2}I_p(x)\right) - \sin\left(\frac{k\pi}{2}I_p(y)\right)}{I_p(x) - I_p(y)}$$

Proof. The chord of $I_p(u)$ with endpoints y and x is given by

$$f(u) = \frac{I_p(x) - I_p(y)}{x-y}(u-x) + I_p(x). \tag{A.2}$$

Lemma A.1 implies, $I_p(u) < f(u)$ for any $p > 1$ and $u \in (0,1)$. Hence and by virtue of the hypothesis,

$$\int_y^x \cos\left(\frac{k\pi}{2}I_p(u)\right)\mathrm{d}u > \int_y^x \cos\left(\frac{k\pi}{2}f(u)\right)\mathrm{d}u = \mathcal{I}_{k,p}(y,x).$$

□

For $0 < s < t < 1$ consider the function

$$G(s,t) := \frac{(1-s^p)^{\frac{1}{p}} I_p(s) - (1-t^p)^{\frac{1}{p}} I_p(t) + \frac{2}{\pi_p}(t-s)}{(1-s^p)^{\frac{1}{p}} - (1-t^p)^{\frac{1}{p}}}. \tag{A.3}$$

Lemma A.3. *Let $k \in \mathbb{N}$ be fixed. Let $0 \le s < t \le 1$ be such that $\cos\left(\frac{k\pi}{2}I_p(u)\right)$ is increasing for all $u \in [s,t]$. Then*

$$\int_s^t \cos\left(\frac{k\pi}{2}I_p(u)\right)\mathrm{d}u > \mathcal{J}^{(1)}_{k,p}(s,t) + \mathcal{J}^{(2)}_{k,p}(s,t)$$

where

$$\begin{aligned}
\mathcal{J}^{(1)}_{k,p}(s,t) &:= \frac{\pi_p}{k\pi}(1-s^p)^{\frac{1}{p}}\left[\sin\left(\frac{k\pi}{2}G(s,t)\right) - \sin\left(\frac{k\pi}{2}I_p(s)\right)\right], \\
\mathcal{J}^{(2)}_{k,p}(s,t) &:= \frac{\pi_p}{k\pi}(1-t^p)^{\frac{1}{p}}\left[\sin\left(\frac{k\pi}{2}I_p(t)\right) - \sin\left(\frac{k\pi}{2}G(s,t)\right)\right].
\end{aligned}$$

Proof. The tangent to the curve $I_p(u)$ at any point $u = s$ is given by

$$\gamma_s(u) = \frac{2}{\pi_p(1-s^p)^{\frac{1}{p}}}(u-s) + I_p(s). \tag{A.4}$$

By virtue of Lemma A.1, for any $p > 1$ and $u \in [0,1]$, we have $I_p(u) > \gamma_s(u)$. The intersection point y of the tangents to $I_p(u)$ at s and t is then given by

$$y = \frac{\frac{\pi_p}{2}(1-t^p)^{\frac{1}{p}}(1-s^p)^{\frac{1}{p}}[I_p(s) - I_p(t)] + t(1-s^p)^{\frac{1}{p}} - s(1-t^p)^{\frac{1}{p}}}{(1-s^p)^{\frac{1}{p}} - (1-t^p)^{\frac{1}{p}}}.$$

Moreover, $\gamma_s(y) = \gamma_t(y) = G(s,t)$.

Then, because of the hypothesis,

$$\int_s^t \cos\left(\frac{k\pi}{2} I_p(u)\right) \mathrm{d}x > \int_s^y \cos\left(\frac{k\pi}{2}\gamma_s(u)\right) \mathrm{d}u + \int_y^t \cos\left(\frac{k\pi}{2}\gamma_t(u)\right) \mathrm{d}u,$$

where

$$\int_s^y \cos\left(\frac{k\pi}{2}\gamma_s(u)\right) \mathrm{d}u = \mathcal{J}_{k,p}^{(1)}(s,t)$$

and

$$\int_y^t \cos\left(\frac{k\pi}{2}\gamma_t(u)\right) \mathrm{d}u = \mathcal{J}_{k,p}^{(2)}(s,t).$$

The claimed assertion follows. $\square$

A.2 Estimates for $a_k(p)$ when $k \equiv_4 3$

For $k = 4j-1$, $j \in \mathbb{N}$. The integrand $\cos\left(\frac{k\pi}{2} I_p(u)\right)$ in (5.1) for $u \in [0,1]$ is monotonically decreasing in j disjoint segments

$$[\tilde{y}_i, \tilde{x}_i] \qquad i = 1, \ldots, j$$

and it is monotonically increasing in j disjoint segments

$$[\tilde{s}_i, \tilde{t}_i] \qquad i = 1, \ldots, j,$$

so that

$$[0,1] = \left(\bigcup_{i=1}^{j}[\tilde{y}_i, \tilde{x}_i]\right) \cup \left(\bigcup_{i=1}^{j}[\tilde{s}_i, \tilde{t}_i]\right)$$

where $\tilde{y}_1 = 0$, $\tilde{t}_j = 1$, $\tilde{s}_i = \tilde{x}_i$ and $\tilde{y}_{i+1} = \tilde{t}_i$. The minimum turning points are such that

$$I_p(\tilde{x}_i) = \frac{4m-2}{k} \qquad \text{for } m = 1, \dots, j$$

and the maximum turning points are such that

$$I_p(\tilde{t}_i) = \frac{4m}{k} \qquad \text{for } m = 1, \dots, j-1$$

We partition each one of these segments into sets of quadrature points as follows. Let $\{m_i^-\}_{i=1}^j$, $\{m_i^+\}_{i=1}^j \subset \mathbb{N} \setminus \{1\}$. Set

$$\begin{aligned}
&x_0 = \tilde{y}_1 = 0, \quad x_{m_1^-} = \tilde{x}_1, \quad x_{1+m_1^-} = \tilde{y}_2, \\
&x_{\sum_{\ell=1}^i m_\ell^-} = \tilde{x}_i, \quad x_{1+\sum_{\ell=1}^i m_\ell^-} = \tilde{y}_{i+1}, && (i = 2, \dots, j) \\
&t_1 = \tilde{s}_1, \quad t_{m_1^+} = \tilde{t}_1, \quad t_{1+m_1^+} = \tilde{s}_2, \\
&t_{\sum_{\ell=1}^i m_\ell^+} = \tilde{t}_i, \quad t_{1+\sum_{\ell=1}^i m_\ell^+} = \tilde{s}_{i+1}, && (i = 2, \dots, j) \\
&t_{\sum_{\ell=1}^j m_\ell^+} = \tilde{t}_j = 1.
\end{aligned}$$

We consider increasing sequences

$$\begin{aligned}
&0 \leq \cdots < x_{m-1} < x_m < \cdots < 1 && (m = 1, \dots, \sum_{\ell=1}^j m_\ell^-) \\
&0 < \cdots < t_{m-1} < t_m < \cdots \leq 1 && (m = 2, \dots, \sum_{\ell=1}^j m_\ell^+)
\end{aligned}$$

such that

$$\left\{x_{1+\sum_{\ell=1}^{i-1} m_\ell^-} < \cdots < x_{\sum_{\ell=1}^i m_\ell^-}\right\} \subset [\tilde{y}_i, \tilde{x}_i] \qquad \text{and}$$
$$\left\{t_{1+\sum_{\ell=1}^{i-1} m_\ell^+} < \cdots < t_{\sum_{\ell=1}^i m_\ell^+}\right\} \subset [\tilde{s}_i, \tilde{t}_i].$$

Lemma A.4. *Let $p > 1$ and $k = 4j - 1$ where $j \in \mathbb{N}$. Then, for $k > 3$*

$$
\begin{aligned}
a_k(p) > \frac{4}{k\pi} \Bigg[& \sum_{m=1}^{m_1^-} \mathcal{I}_{k,p}(x_{m-1}, x_m) + \sum_{\ell=1}^{j-1} \sum_{m=\sum_{i=1}^{\ell} m_i^- + 2}^{\sum_{i=1}^{\ell+1} m_i^-} \mathcal{I}_{k,p}(x_{m-1}, x_m) \\
& + \sum_{m=2}^{m_1^+} \Big(\mathcal{J}_{k,p}^{(1)}(t_{m-1}, t_m) + \mathcal{J}_{k,p}^{(2)}(t_{m-1}, t_m) \Big) \\
& + \sum_{\ell=1}^{j-2} \sum_{m=\sum_{i=1}^{\ell} m_i^+ + 2}^{\sum_{i=1}^{\ell+1} m_i^+} \Big(\mathcal{J}_{k,p}^{(1)}(t_{m-1}, t_m) + \mathcal{J}_{k,p}^{(2)}(t_{m-1}, t_m) \Big) \\
& + \sum_{m=\sum_{i=1}^{j-1} m_i^+ + 2}^{\sum_{i=1}^{j} m_i^+ - 1} \Big(\mathcal{J}_{k,p}^{(1)}(t_{m-1}, t_m) + \mathcal{J}_{k,p}^{(2)}(t_{m-1}, t_m) \Big) \\
& + \mathcal{J}_{k,p}^{(1)}(t_{\sum_{\ell=1}^{j} m_\ell^+ - 1}, 1) \Bigg] .
\end{aligned}
$$

While for $k = 3$, the upper limit of the summation in the third term on the right side of the inequality is $m = m_1^+ - 1$.

Proof. The proof follows from the properties established in lemmas A.2 and A.3 by taking as endpoints $y = x_{m-1}$ and $x = x_m$ for $m = 1, \dots, m_1^-$ and $m = \sum_{i=1}^{\ell} m_i^- + 2, \dots, \sum_{i=1}^{\ell+1} m_i^-$ when $\ell = 1, \dots, j-1$ in the former case and $s = t_{m-1}$ and $t = t_m$ in the latter case for $m = 2, \dots, m_1^+$ and $m = \sum_{i=1}^{\ell} m_i^+ + 2, \dots, \sum_{i=1}^{\ell+1} m_i^+$ when $\ell = 1, \dots, j-1$.

Let $G(s,t)$ be given by the expression (A.3). Observe that for any $j \in \mathbb{N}$ the tangent to the curve $I_p(t)$ at $t_{\sum_{\ell=1}^{j} m_\ell^+} = 1$ is the vertical line $t = 1$ which meets the tangent line at $t_{\sum_{\ell=1}^{j} m_\ell^+ - 1}$ at the point $\Big(1, G(t_{\sum_{\ell=1}^{j} m_\ell^+ - 1}, 1)\Big)$. Moreover,

$$
I_p(t) \geq \gamma_{t_{\sum_{\ell=1}^{j} m_\ell^+ - 1}}(t) \qquad \text{for} \qquad t \in [t_{\sum_{\ell=1}^{j} m_\ell^+ - 1}, 1]
$$

where γ_s is given by (A.4). Hence,

$$
\int_{t_{\sum_{\ell=1}^{j} m_\ell^+ - 1}}^{1} \cos\left(\frac{k\pi}{2} I_p(t) \right) \mathrm{d}t > \mathcal{J}_{k,p}^{(1)}(t_{\sum_{\ell=1}^{j} m_\ell^+ - 1}, 1).
$$

and the proof is complete. ∎

A.3 Estimates for $a_k(p)$ when $k \equiv_4 1$

For $k = 4j-3$, $j \in \mathbb{N}$. The function $\cos\left(\frac{k\pi}{2} I_p(u)\right)$ with $u \in [0,1]$ is monotonically decreasing in j disjoint segments

$$[\tilde{y}_i, \tilde{x}_i] \qquad i = 1, \ldots, j$$

and it is monotonically increasing in $j-1$ disjoint segments

$$[\tilde{s}_i, \tilde{t}_i] \qquad i = 1, \ldots, j-1,$$

so that

$$[0,1] = \left(\bigcup_{i=1}^{j} [\tilde{y}_i, \tilde{x}_i]\right) \cup \left(\bigcup_{i=1}^{j-1} [\tilde{s}_i, \tilde{t}_i]\right)$$

where $\tilde{y}_1 = 0$, $\tilde{x}_j = 1$, $\tilde{s}_i = \tilde{x}_i$ and $\tilde{y}_{i+1} = \tilde{t}_i$. The minimum turning points are such that

$$I_p(\tilde{x}_i) = \frac{4m-2}{k} \qquad \text{for } m = 1, \ldots, j-1$$

and the maximum turning points are such that

$$I_p(\tilde{t}_i) = \frac{4m}{k} \qquad \text{for } m = 1, \ldots, j-1.$$

We partition each one of these segments into sets of quadrature points as follows. Let $\{m_i^-\}_{i=1}^{j}$, $\{m_i^+\}_{i=1}^{j-1} \subset \mathbb{N} \setminus \{1\}$. Set

$$x_0 = \tilde{y}_1 = 0, \quad x_{m_1^-} = \tilde{x}_1, \quad x_{1+m_1^-} = \tilde{y}_2,$$

$$x_{\sum_{\ell=1}^{i} m_\ell^-} = \tilde{x}_i, \qquad (i = 2, \ldots, j)$$

$$x_{1+\sum_{\ell=1}^{i} m_\ell^-} = \tilde{y}_{i+1}, \qquad (i = 2, \ldots, j-1)$$

$$x_{\sum_{\ell=1}^{j} m_\ell^-} = \tilde{x}_j = 1,$$

$$t_1 = \tilde{s}_1, \quad t_{m_1^+} = \tilde{t}_1, \quad t_{1+m_1^+} = \tilde{s}_2,$$

$$t_{\sum_{\ell=1}^{i} m_\ell^+} = \tilde{t}_i, \qquad (i = 2, \ldots, j-1)$$

$$t_{1+\sum_{\ell=1}^{i} m_\ell^+} = \tilde{s}_{i+1}, \qquad (i = 2, \ldots, j-2)$$

We consider an increasing sequence of quadrature points

$$0 \leq \dots < x_{m-1} < x_m < \dots \leq 1 \qquad\qquad (m = 1, \dots, \sum_{\ell=1}^{j} m_\ell^-)$$
$$0 < \dots < t_{m-1} < t_m < \dots < 1 \qquad\qquad (m = 2, \dots, \sum_{\ell=1}^{j-1} m_\ell^+)$$

such that

$$\left\{ x_{1+\sum_{\ell=1}^{i-1} m_\ell^-} < \dots < x_{\sum_{\ell=1}^{i} m_\ell^-} \right\} \subset [\tilde{y}_i, \tilde{x}_i] \qquad \text{and}$$
$$\left\{ t_{1+\sum_{\ell=1}^{i-1} m_\ell^+} < \dots < t_{\sum_{\ell=1}^{i} m_\ell^+} \right\} \subset [\tilde{s}_i, \tilde{t}_i].$$

Lemma A.5. *Let $p > 1$ and $k = 4j - 3$ where $j \in \mathbb{N}$. Then, for $k > 1$*

$$a_k(p) > \frac{4}{k\pi} \left[\sum_{m=1}^{m_1^-} \mathcal{I}_{k,p}(x_{m-1}, x_m) + \sum_{\ell=1}^{j-1} \sum_{m=\sum_{i=1}^{\ell} m_i^- + 2}^{\sum_{i=1}^{\ell+1} m_i^-} \mathcal{I}_{k,p}(x_{m-1}, x_m) + \right.$$
$$+ \sum_{m=2}^{m_1^+} \left(\mathcal{J}_{k,p}^{(1)}(t_{m-1}, t_m) + \mathcal{J}_{k,p}^{(2)}(t_{m-1}, t_m) \right)$$
$$\left. + \sum_{\ell=1}^{j-2} \sum_{m=\sum_{i=1}^{\ell} m_i^+ + 2}^{\sum_{i=1}^{\ell+1} m_i^+} \left(\mathcal{J}_{k,p}^{(1)}(t_{m-1}, t_m) + \mathcal{J}_{k,p}^{(2)}(t_{m-1}, t_m) \right) \right].$$

While for $k = 1$, the right side of the inequality includes only the first summation.

Proof. The proof follows directly from lemmas A.2 and A.3 by taking as endpoints $y = x_{m-1}$ and $x = x_m$ for $m = 1, \dots, m_1^-$ and $m = \sum_{i=1}^{\ell} m_i^- + 2, \dots, \sum_{i=1}^{\ell+1} m_i^-$ when $\ell = 1, \dots, j-1$ in the former case and $s = t_{m-1}$ and $t = t_m$ in the latter case for $m = 2, \dots, m_1^+$ and $m = \sum_{i=1}^{\ell} m_i^+ + 2, \dots, \sum_{i=1}^{\ell+1} m_i^+$ when $\ell = 1, \dots, j-2$. $\square$

Appendix B

Appendix

B.1 Matlab Codes

Here we include some Matlab codes which provide lower bound estimates for the sine Fourier coefficients $a_j(p)$ of the $\sin_p(\pi_p x) \in L^r(0,1)$ and $r \in (1,\infty)$ when $j = 3$ and 9. These codes are based on the calculations performed in Appendix A and the results they provide are presented in the Tables 5.1 and 5.2, respectively.

B.1.1 Lower bound estimates for $a_3(p)$, $p > 1$

```
% Lower bound estimates for a3(p) when p>1:
% x1, t1, t2: vectors of linearly spaced quadratures.
% components of x1 divide the interval where the integrand
    of a3(p) is decreasing.
% components of t1 and t2 divide the interval where the
    integrand is increasing.
function a3=a3lowerbound(p,x1,t1,t2)
pip=2*pi/p/sin(pi/p);
I1(1)=0;
for j=2:length(x1)
I1(j)=I1(j-1)+8/9/pi^2*(x1(j)-x1(j-1))*(sin(3*pi/2*betainc(
    x1(j)^p,1/p,1-1/p))-sin(3*pi/2*betainc(x1(j-1)^p,1/p,1-1/
```

```
p)))/(betainc(x1(j)^p,1/p,1-1/p)-betainc(x1(j-1)^p,1/p,1-1/p));
end
lI1=I1(length(x1))
h1(1)=0;
for j=2:length(t1)
    g1(j)=((1-t1(j)^p)^(-1/p)*betainc(t1(j-1)^p,1/p,1-1/p)-(1-t1(j-1)^p)^(-1/p)*betainc(t1(j)^p,1/p,1-1/p)+2/pip*(1-t1(j-1)^p)^(-1/p)*(1-t1(j)^p)^(-1/p)*(t1(j)-t1(j-1)))/((1-t1(j)^p)^(-1/p)-(1-t1(j-1)^p)^(-1/p));
h1(j)=h1(j-1)+4*pip/9/pi^2*((1-t1(j)^p)^(1/p)*sin(3*pi/2*betainc(t1(j)^p,1/p,1-1/p))-(1-t1(j-1)^p)^(1/p)*sin(3*pi/2*betainc(t1(j-1)^p,1/p,1-1/p))+sin(3*pi/2*g1(j))*((1-t1(j-1)^p)^(1/p)-(1-t1(j)^p)^(1/p)));
end
lh1=h1(length(t1))
lh2=4/9/pi^2*pip*(1-t2^p)^(1/p)*(sin(3*pi/2*(2/pip/(1-t2^p)^(1/p)*(1-t2)+betainc(t2^p, 1/p, 1-1/p)))-sin(3*pi/2*betainc(t2^p,1/p,1-1/p)));
a3=lI1+lh1+lh2

% Example(1):
% >>p=1.5, z1=fzero(@(x) betainc(x.^p,1./p,(p-1)./p)-2/3,0.9); %the x-coordinate of the minimum point of the integrand of a3(p).
%
% >>x1=linspace(0,z1,3);
% >>t11=linspace(z1,1,3);
% >>t1=[t11(1) t11(2)];
% >>t2=t11(2);
```

```
% >>a3lowerbound(1.5,x1,t1,t2)
% ans=
%       0.069231965372217
%
% Example(2):
% >>p=1.5, z1=fzero(@(x) betainc(x.^p,1./p,(p-1)./p)-2/3,0.9); %the x-coordinate of the minimum point of the integrand of a3(p).
%
% >>x1=linspace(0,z1,4);
% >>t11=linspace(z1,1,3);
% >>t1=[t11(1) t11(2)];
% >>t2=t11(2);
% >>a3lowerbound(1.5,x1,t1,t2)
% ans=
%       0.091292084421940
%
% Example(3):
% >>p=1.9, z1=fzero(@(x) betainc(x.^p,1./p,(p-1)./p)-2/3,0.9); %the x-coordinate of the minimum point of the integrand of a3(p).
%
% >>x1=linspace(0,z1,4);
% >>t11=linspace(z1,1,3);
% >>t1=[t11(1) t11(2)];
% >>t2=t11(2);
% >>a3lowerbound(1.9,x1,t1,t2)
% ans
%    =0.005348565159729
```

B.1.2 Lower bound estimates for $a_9(p)$, $p > 1$

```
% Lower bound estimates for a9(p) when p>1:
% x1,t1,x2,t2,x3: vectors of linearly spaced quadratures.
% components of x1,x2,x3 divide the intervals where the integrand of a9(p) is decreasing.
% components of t1 and t2 divide the intervals where the integrand is increasing.
function a9=a9lowerbound(p,x1,t1,x2,t2,x3)
pip=2*pi/p/sin(pi/p);
I1(1)=0;
for j=2:length(x1)
I1(j)=I1(j-1)+8/81/pi^2*(x1(j)-x1(j-1))*(sin(9*pi/2*betainc(x1(j)^p,1/p,1-1/p))-sin(9*pi/2*betainc(x1(j-1)^p,1/p,1-1/p)))/(betainc(x1(j)^p,1/p,1-1/p)-betainc(x1(j-1)^p,1/p,1-1/p));
end
lI1=I1(length(x1))
I2(1)=0;
for j=2:length(x2)
I2(j)=I2(j-1)+8/81/pi^2*(x2(j)-x2(j-1))*(sin(9*pi/2*betainc(x2(j)^p,1/p,1-1/p))-sin(9*pi/2*betainc(x2(j-1)^p,1/p,1-1/p)))/(betainc(x2(j)^p,1/p,1-1/p)-betainc(x2(j-1)^p,1/p,1-1/p));
end
lI2=I2(length(x2))
I3(1)=0;
for j=2:length(x3)
I3(j)=I3(j-1)+8/81/pi^2*(x3(j)-x3(j-1))*(sin(9*pi/2*betainc(x3(j)^p,1/p,1-1/p))-sin(9*pi/2*betainc(x3(j-1)^p,1/p,1-1/p)))/(betainc(x3(j)^p,1/p,1-1/p)-betainc(x3(j-1)^p,1/p
```

```
,1-1/p));
end
lI3=I3(length(x3))
h1(1)=0;
for j=2:length(t1)
    g1(j)=((1-t1(j)^p)^(-1/p)*betainc(t1(j-1)^p,1/p,1-1/p)
        -(1-t1(j-1)^p)^(-1/p)*betainc(t1(j)^p,1/p,1-1/p)+2/
        pip*(1-t1(j-1)^p)^(-1/p)*(1-t1(j)^p)^(-1/p)*(t1(j)-t1
        (j-1)))/((1-t1(j)^p)^(-1/p)-(1-t1(j-1)^p)^(-1/p));
h1(j)=h1(j-1)+4*pip/81/pi^2*((1-t1(j)^p)^(1/p)*sin(9*pi/2*
    betainc(t1(j)^p,1/p,1-1/p))-(1-t1(j-1)^p)^(1/p)*sin(9*pi
    /2*betainc(t1(j-1)^p,1/p,1-1/p))+sin(9*pi/2*g1(j))*((1-t1
    (j-1)^p)^(1/p)-(1-t1(j)^p)^(1/p)));
end
lh1=h1(length(t1))
h2(1)=0;
for j=2:length(t2)
    g2(j)=((1-t2(j)^p)^(-1/p)*betainc(t2(j-1)^p,1/p,1-1/p)
        -(1-t2(j-1)^p)^(-1/p)*betainc(t2(j)^p,1/p,1-1/p)+2/
        pip*(1-t2(j-1)^p)^(-1/p)*(1-t2(j)^p)^(-1/p)*(t2(j)-t2
        (j-1)))/((1-t2(j)^p)^(-1/p)-(1-t2(j-1)^p)^(-1/p));
h2(j)=h2(j-1)+4*pip/81/pi^2*((1-t2(j)^p)^(1/p)*sin(9*pi/2*
    betainc(t2(j)^p,1/p,1-1/p))-(1-t2(j-1)^p)^(1/p)*sin(9*pi
    /2*betainc(t2(j-1)^p,1/p,1-1/p))+sin(9*pi/2*g2(j))*((1-t2
    (j-1)^p)^(1/p)-(1-t2(j)^p)^(1/p)));
end
lh2=h2(length(t2))
a9=lI1+lI2+lI3+lh1+lh2
```

```
% Example(1):
% z1, z3 are the x-coordinates of the minimum points of the
    integrand of a9(p).
% >>p=1.5, z1=fzero(@(x) betainc(x.^p,1./p,(p-1)./p)
    -2/9,0.4)
% >>z3=fzero(@(x) betainc(x.^p,1./p,(p-1)./p)-6/9,0.9)
%
% z2, z4 are the x-coordinates of the maximum points of the
    integrand of a9(p).
% >>z2=fzero(@(x) betainc(x.^p,1./p,(p-1)./p)-4/9,0.8)
% >>z4=fzero(@(x) betainc(x.^p,1./p,(p-1)./p)-8/9,1)
%
% >>x1=linspace(0,z1,5);
% >>t1=linspace(z1,z2,5);
% >>x2=linspace(z2,z3,5);
% >>t2=linspace(z3,z4,4);
% >>x3=linspace(z4,1,2);
% >>a9lowerbound(1.5,x1,t1,x2,t2,x3)
% ans=
%      8.768808382550713e-06
```

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